$$y = e^{-x}(A\cos 2x + B\sin 2x) + \underset{=0.16}{c}\,e^{0.5x} + \underset{=0}{k}\cos 10x + \underset{=2}{M}\sin 10x$$

$$= e^{-x}(A\cos 2x + B\sin 2x) + 0.16e^{0.5x} + 2\sin 10x$$

$$y' = -\underset{=1}{e^{-x}}A\underset{=1}{\cos 2x} - 2Ae^{-x}\underset{=0}{\sin 2x} - e^{-x}B\underset{=0}{\sin 2x} + 2B\underset{=1}{e^{-x}}\underset{=1}{\cos 2x} + 0.08\underset{=1}{e^{0.5x}} + 20\underset{=1}{\cos 10x}$$

初始值 $y(0) = A + 0.16 = 0.16.\quad A = 0.$

$$y'(0) = -A + 2B + 0.08 + 20 = 40.08.$$

$$= 2B + 20.08 = 40.08$$

$$B = 10.$$

$$y = e^{-x}\cdot 10\sin 2x + 0.16e^{0.5x} + 2\sin 10x \quad \#$$

冪级數

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$\boxed{\begin{aligned}
&\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad |x|<1 \\
&e^x = \sum_{m=0}^{\infty}\frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&\cos x = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots \\
&\sin x = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots
\end{aligned}}$$

例：解 $y'-y=0$

(sol) $y=a_0+a_1x^1+a_2x^2+a_3x^3+\cdots\cdots$

$y'=a_1+2a_2x+3a_3x^2+\cdots\cdots$

代回 $\underbrace{(a_1+2a_2x+3a_3x^2+\cdots\cdots)}_{y'}-\underbrace{(a_0+a_1x^1+a_2x^2+a_3x^3+\cdots\cdots)}_{y}=0$

$(a_1-a_0)+(2a_2-a_1)x+(3a_3-a_2)x^2+\cdots=0$

$\rightarrow\ a_1=a_0,\quad 2a_2=a_1,\quad 3a_3=a_2$

$\rightarrow\ a_2=\frac{a_1}{2},\quad a_3=\frac{a_2}{3}=\frac{a_0}{3\times 2!}=\frac{a_0}{3!}$

$y=a_0+a_0x+\frac{a_0}{2!}x^2+\frac{a_0}{3!}x^3+\cdots\cdots$

$=a_0(\underbrace{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots}_{e^x})=a_0e^x$

可分離O.D.E

$y'-y=0$

$\int\frac{dy}{y}=\int 1\cdot dx$

$e^{\ln|y|}=e^{x+\tilde{c}}\quad[e^{\tilde{c}}=c]$

$y=c\cdot e^x$

例：求 $y'=2xy$

(sol) $y=\sum_{m=0}^{\infty}a_mx^m=a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots$

$y'=\sum_{m=1}^{\infty}m\cdot a_mx^{m-1}=a_1+2a_2x+3a_3x^2+\cdots\cdots$

代回 $\underbrace{a_1+2a_2x+3a_3x^2+\cdots\cdots}_{y'}=2x(\underbrace{a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots}_{y})$

$=0+2a_0x+2a_1x^2+2a_2x^3+2a_3x^4+\cdots\cdots$

左右比較 $a_1=0,\quad 2a_2=2a_0,\quad 3a_3=2\underset{=0}{a_1},\quad 4a_4=2a_2,\quad 5a_5=2\underset{=0}{a_3}$

因 $a_1=0$, $\therefore a_3=0,\ a_5=0$

偶數項 $a_2=a_0,\quad a_4=\frac{a_2}{2}=\frac{a_0}{2!},\quad a_6=\frac{a_4}{3}=\frac{a_0}{3!}$

故 $y=a_0(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots\cdots)$

令 $x^2=u$

$(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots)$

$e^u=e^{x^2}$

$\therefore y=a_0e^{x^2}$ #

$\int\frac{1}{y}dy=\int 2x\cdot dx$

$e^{\ln|y|}=e^{x^2+\tilde{c}}$

$y=ce^{x^2}\quad(c=a_0)$

例: 求 $y''+y=0$

(sol) $y=\sum_{m=0}^{\infty} a_m x^m = a_0+a_1x+a_2x^2+\cdots\cdots$

$y''=\sum_{m=2}^{\infty} m\cdot(m-1)a_m x^{m-2} = 2a_2+(3\cdot 2a_3x)+(4\cdot 3a_4x^2)+\cdots\cdots$

微分二次.失去 a_0, a_1

代回 $(2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+\cdots)+(a_0+a_1x+a_2x^2+\cdots\cdots)=0$

同冪數之x搜集整理,得

$(2a_2+a_0)+(3\cdot 2a_3+a_1)x+(4\cdot 3a_4+a_2)x^2+\cdots\cdots=0$

$\rightarrow\ 2a_2+a_0=0,\quad 3\cdot 2a_3+a_1=0,\quad 4\cdot 3a_4+a_2=0$

分類後可得 $a_2, a_4\cdots$ 以 a_0 表示
$a_3, a_5\cdots\cdots$ 以 a_1 表示

$a_2=-\frac{a_0}{2!},\quad a_3=-\frac{a_1}{3!},\quad a_4=-\frac{a_2}{4\cdot 3}=\frac{a_0}{4!}$

$$y=a_0+a_1x-\frac{a_0}{2!}x^2-\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5-\cdots\cdots$$

$$=a_0\underbrace{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-+\cdots\right)}_{\cos x}+a_1\underbrace{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots\right)}_{\sin x}$$

$$=a_0\cos x+a_1\sin x \ \#$$

$y''+y=0$
$\lambda^2+1=0$
$\lambda=0\pm i$
$y=e^{-\frac{a}{2}x}(A\cos x+B\sin x)$
$a=0,\ e^0=1$

$|x-x_0|<R \Rightarrow$ 收斂.

$|x-x_0|>R \Rightarrow$ 發散.

$R=1\Big/\lim_{m\to\infty}\sqrt[m]{|a_m|}$

$R=1\Big/\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|$ 第m+2項值比第m+1項小,即越大項值越小 $\rightarrow$ 收斂.

例：級數為 $\sum_{m=0}^{\infty} \underbrace{m!}_{a_m} x^m = 1+x+2x^2+6x^3$

(sol) $a_m = m!$

$$\frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = (m+1) \qquad m\to\infty,\ (m+1)\to\infty$$

故除了 $x=0$ 外，級數發散。

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\infty} = 0$$ 沒有收斂半徑，發散

例： 級數 $\frac{1}{1-x} = \sum_{m=0}^{\infty} \underbrace{\ }_{a_m} x^m = 1+x+x^2+\cdots \quad (|x|<1)$

(sol) $a_m = 1$, for all m

$$R = \frac{1}{\lim_{m\to\infty}\underbrace{\left|\frac{a_{m+1}}{a_m}\right|}_{=1}} = 1$$

故 $|x-\underbrace{x_0}_{=0}|<R$. $|x|<1$ 區間內，級數收斂。

例： 級數 $e^x = \sum_{m=0}^{\infty} \underbrace{\frac{x^m}{m!}}_{a_m} = 1+x+\frac{x^2}{2!}+\cdots$

(sol) $a_m = \frac{1}{m!}$

$$\frac{a_{m+1}}{a_m} = \frac{\frac{1}{(m+1)!}}{\frac{1}{m!}} = \frac{1}{m+1} \to 0 \text{ 當 } m\to\infty$$

$$R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\lim\left|\frac{1}{m+1}\right|} = \lim_{m\to\infty}(m+1) \to \infty$$

$|x-x_0|<R$，而 $|x|<\infty$，對所有 x 都可收斂

例：求下列級數的收斂半徑 $\sum_{m=0}^{\infty} \underbrace{\frac{(-1)^m}{8^m}}_{=a_m} x^{3m} = 1-\frac{x^3}{8}+\frac{x^6}{64}-\frac{x^9}{512}+-\cdots$

此級數以 $t=x^3$ 之冪次方展開，係數為 $a_m = \frac{(-1)^m}{8^m}$

$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{8^m}{8^{m+1}} = \frac{1}{8}$$

$\therefore R=8$，所以 $|t| = |x^3| < 8$，即 $|x|<2$ 時本級數收斂。