Laplace転換.

$$\boxed{F_s = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt}$$

$f(t) = \mathcal{L}^{-1}(F)$

$f(t) = F(s)$ 　原函数変数為t, 転換式的変数為s.

$y(t) = Y(s)$

重要

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
$\mathcal{L}\{1\}$	$\frac{1}{s}$	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$\mathcal{L}\{f(t)e^{at}\}$	$F(s-a)$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

例: 使 $f(t)=1$ 　$(t \geq 0)$ 求 $F(s)$

(sol) $\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st} \cdot 1 \cdot dt$ 　　$\left(\frac{1}{-s}\right) d(-st)$ 　$\div(-s)$

$\int e^{ax}dx = \frac{1}{a}e^{ax} + c$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= -\left(\frac{1}{s} \cdot \underbrace{\cancel{\frac{1}{e^\infty}}}_{0} - \frac{1}{s} \cdot \underbrace{\cancel{\frac{1}{e^0}}}_{1}\right) = \frac{1}{s}$$ 　　$e^0 = 1$

例: 使 $f(t) = e^{at}$ 　$(t \geq 0)$ 　a為一常数. 求 $F(s)$

(sol) $\mathcal{L}(f) = \mathcal{L}(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$

$e^{-st} \cdot e^{at}$

$= e^{-(s-a)t}$

$= \frac{1}{-(s-a)} e^{-(s-a)t} \, d(-(s-a)t)$

$\div -(s-a)$

$$= -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty = -\frac{1}{s-a}\left(\underbrace{\cancel{\frac{1}{e^{(s-a)\infty}}}}_{0} - \underbrace{\cancel{\frac{1}{e^{(s-a)0}}}}_{1}\right)$$

$$= \frac{1}{s-a} \qquad (s>a)$$

$$\boxed{\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]}$$ a.b 為任意常數

例：使 $f(t)=\cosh at = \frac{1}{2}(e^{at}+e^{-at})$ 求 $\mathcal{L}(f)$

(sol) $\mathcal{L}(\cosh at) = \mathcal{L}[\frac{1}{2}(e^{at}+e^{-at})] = \frac{1}{2}\mathcal{L}(e^{at})+\frac{1}{2}\mathcal{L}(e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \quad \#$$ $(s>a,\ a\geq 0)$

例：使 $f(t)=\sinh at = \frac{1}{2}\mathcal{L}(e^{at})-\frac{1}{2}\mathcal{L}(e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right) = \frac{a}{s^2+a^2}$$ $(s>a\geq 0)$

$$\frac{1}{2}\frac{(s+a)-(s-a)}{(s-a)(s+a)} = \frac{2a}{s^2-a^2}\cdot\frac{1}{2} = \frac{a}{s^2-a^2}$$

例：$f(t)=e^{i\omega t}$ 求 $\mathcal{L}(f)$

(sol) $\mathcal{L}(f) = \int_0^\infty e^{-st}e^{i\omega t}dt$

$$= \int_0^\infty \underset{\text{除}-(s-i\omega)}{e^{-(s-i\omega)t}}\, \underset{\text{乘}-(s-i\omega)}{dt} = -\frac{1}{(s-i\omega)}e^{-(s-i\omega)t}\Big|_0^\infty$$

(為了讓 $-(s-i\omega)t$ 可以積分)

$$= -\frac{1}{(s-i\omega)}\left(\underset{\to 0}{\frac{1}{e^{(s-i\omega)\infty}}}-\underset{\to 1}{\frac{1}{e^0}}\right)$$

$$= \frac{1}{s-i\omega} = \frac{(s+i\omega)}{(s-i\omega)(s+i\omega)}$$ (上下通乘 $s+i\omega$)

$$= \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}+i\frac{\omega}{s^2+\omega^2} \quad ※$$

上題在 Euler formula

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos\omega t+i\sin\omega t)$$

$$= \mathcal{L}(\cos\omega t)+i\mathcal{L}(\sin\omega t) = \frac{s}{s^2+\omega^2}+i\frac{\omega}{s^2+\omega^2}$$

6-2. 微分或積分的転換

定理：若 $f(t)$ 為連續函数 $(t\geq 0)$ $f'(t)$ 是分段連續，則

重要. $\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0)}$ —— ①

$\mathcal{L}(f'') = \mathcal{L}[(f')'] = s[\mathcal{L}(f')] - f'(0)$

重要 $\boxed{\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)}$ —— ②

同理 $\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2f(0) - sf'(0) - f''(0)$ —— ③

重要 $\mathcal{L}(f^n) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$ —— ④

例：$f(t) = \sin^2 t$，$f(0) = 0$

$f'(t) = 2\sin t\cdot\cos t = \sin 2t$，$f'(0) = 0$.

$\frac{d\sin^2 t}{d\sin t}\cdot\frac{d\sin t}{dt} = 2\sin t\cdot\cos t$

求 $\mathcal{L}(f)$

(sol) $\mathcal{L}(f') = \mathcal{L}(\sin 2t) = s\mathcal{L}(f) - \cancel{f(0)}^{0}$

$\omega = 2,\ \frac{2}{s^2+4}$

$= s\mathcal{L}(\sin^2 t)$

$\mathcal{L}(\sin^2 t) = \frac{2}{s(s^2+4)}$

$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$

$\omega = 2.$ $\mathcal{L}(\sin 2t) = \frac{2}{s^2+2^2}$

例：求 $\mathcal{L}(\cos^2 t)$

(sol) $f(t) = \cos^2 t$

$f(0) = 1$

$f'(t) = \frac{d\cdot\cos^2 t}{d\cos t}\cdot\frac{d\cos t}{dt} = 2\cos t(-\sin t)$

$= -2\cos t\cdot\sin t = -\sin 2t$

$f'(0) = 0.$

$\mathcal{L}(f') = \mathcal{L}(-\sin 2t) = -\mathcal{L}(\sin 2t) = -\frac{2}{s^2+4}$

$\hookrightarrow = s\mathcal{L}(f) - \cancel{f(0)}^{1}$

$\mathcal{L}(\cos^2 t) = \frac{1}{s}\left[1 - \frac{2}{s^2+4}\right] = \frac{1}{s}\left[\frac{s^2+2}{s^2+4}\right]$

$\frac{s^2+4-2}{s^2+4} = \frac{s^2+2}{s^2+4}$

例: $f(t)=t\sin\omega t$ 求 $\mathcal{L}(f)$

(sol) $f(0)=0$.

$$f'(t)=(t\sin\omega t)'=\sin\omega t+t(\sin\omega t)'$$
$$=\sin\omega t+t\omega\cos\omega t$$

$f'(0)=0$.

由 $\mathcal{L}(f'')=\mathcal{L}[(f')']=\mathcal{L}[(\sin\omega t)'+(t\omega\cos\omega t)']$

$$=\mathcal{L}[\omega\cos\omega t+(\omega\cos\omega t-t\omega\cdot\omega\sin\omega t)]$$
$$=\mathcal{L}[2\omega\cdot\cos\omega t-t\omega^2\sin\omega t]$$
$$=\mathcal{L}[2\omega\cos\omega t-\omega^2(t\sin\omega t)]$$
$$=\mathcal{L}[2\omega\cos\omega t-\omega^2 f(t)]$$

$$=s^2\mathcal{L}(f)-s\cancel{f(0)}-\cancel{f'(0)}$$

$$s^2\mathcal{L}(f)=\mathcal{L}[2\omega\cos\omega t-\omega^2 f(t)]$$

$$(s^2+\omega^2)\mathcal{L}(f)=2\omega\mathcal{L}(\cos\omega t)$$

通除 $(s^2+\omega^2)$ $\mathcal{L}(f)=2\omega\dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}(f)=\mathcal{L}(t\sin\omega t)=\frac{2\omega s}{(s^2+\omega^2)^2}$$ #

例: $f(t)=t^2$ 試求 $\mathcal{L}(f)$

(sol) 因 $f(0)=0$

$$f'(0)=(t^2)'=2t$$
$$f''(t)=(t^2)''=2$$ 已知 $\mathcal{L}(1)=\frac{1}{s}$ 所以微分2次

$$\mathcal{L}(f'')=s^2\mathcal{L}(f)-s\cancel{f(0)}-\cancel{f'(0)}$$
$$=s^2\mathcal{L}(f)$$

$$s^2\mathcal{L}(f)=\frac{2}{s}$$ $\left[\mathcal{L}(2)=2\left(\frac{1}{s}\right)=\frac{2}{s}\right]$

$$\mathcal{L}(f)=\mathcal{L}(t^2)=\frac{2}{s^3}$$ #

⑥ DE

$$y'' + ay' + by = r(t) \quad y(0) = k_0 \quad y'(0) = k_1$$

step 1 Laplace転換.

$Y = \mathcal{L}(y) \quad R = \mathcal{L}(r)$ 得

重要

$$\boxed{\begin{aligned} \mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) \\ \mathcal{L}(y') &= sY - y(0) \\ \mathcal{L}(y) &= Y \end{aligned}}$$

$$\mathcal{L}[y'' + ay' + by] = \mathcal{L}[r(t)]$$

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

收集Y項.

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s)$$

step 2, 对Y代数解

$$Q_s = \frac{1}{s^2 + as + b}$$ (称為転換函数)

$$Y(s) = [(s+a)y(0) + y'(0)]Q_s + R(s)Q(s)$$

如. $y(0) = y'(0) = 0$, 則 $Y = RQ$

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$ $y(t)$ 輸出, $r(t)$ 輸入

step 3 $y(t) = \underbrace{\mathcal{L}^{-1}}_{\text{逆転換}}[Y(s)]$

例: 求 $y'' - y = t \quad y(0) = 1, \; y'(0) = 1$

(sol) step 1 $\mathcal{L}(y'' - y = t)$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\rightarrow \underbrace{s^2Y - s\,\cancel{y(0)}^{1} - \cancel{y'(0)}^{1}}_{y''} - \underbrace{Y}_{y} = \frac{1}{s^2}$$

$$(s^2 - 1)Y = s + 1 + \frac{1}{s^2}$$

step 2 左右除 $(s^2 - 1)$

$$Y = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

$$\sinh at = \frac{a}{s^2 - a^2}$$
$$\downarrow$$
$$\sinh t = \frac{1}{s^2 - 1} \quad (a = 1)$$

step 3 $y(t) = \mathcal{L}^{-1}(Y)$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^t + \sinh t - t \quad ※$$

分解 $$\frac{1}{s^2(s^2-1)} = \frac{A}{s^2} + \frac{B}{s^2-1}$$

$$1 = A(s^2-1) + Bs^2.$$

$$As^2 - A + Bs^2 = 1.$$

$\therefore A = -1.\ B = 1 \cdot (-s^2 - (-1) + s^2 = 1$

代回 $$= \frac{-1}{s^2} + \frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1} - \frac{1}{s^2}$$

例 解 $y''+2y'+y=e^{-t}$, $y(0)=-1$, $y'(0)=1$

(sol)

$\mathcal{L}(e^{at})=\frac{1}{s-a}$

$\mathcal{L}(e^{-t})=\frac{1}{s-(-1)}$

step1: $\mathcal{L}(y''+2y'+y=e^{-t})$

$$\underbrace{[s^2Y-s\overset{-1}{y(0)}-\overset{1}{y'(0)}]}_{y''}+2\underbrace{[sY-\overset{-1}{y(0)}]}_{y'}+\underbrace{Y}_{y}=\frac{1}{s-(-1)}$$

$$(s^2Y+s-1)+2(sY+1)+Y=\frac{1}{s+1}$$

step2: 收集Y項

移項 $(s^2+2s+1)Y=-s-1+\frac{1}{s+1}$

$$(s+1)^2Y=-s-1+\frac{1}{s+1}$$

通除 $(s+1)^2$ $\quad Y=\frac{-s-1}{(s+1)^2}+\frac{1}{(s+1)^3}$

$$\mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\}=-\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}=-\mathcal{L}^{-1}e^{-t}$$

step3: $y=\mathcal{L}^{-1}(Y)$

$$=\mathcal{L}^{-1}\left\{\frac{-(s+1)}{(s+1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{-1}{(s+1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$=-e^{-t}+\frac{1}{(3-1)!}t^{3-1}\cdot e^{-t}$$

$$=-e^{-t}+\frac{1}{2!}t^2\cdot e^{-t}$$

$$=\left(\frac{1}{2}t^2-1\right)e^{-t}$$

#

公式:

① $\mathcal{L}^{-1}[f(s-a)]=e^{at}\cdot f(t)$ 偏移定理

② $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}=\frac{t^n}{n!}$

①,② $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\}=\frac{t^n}{n!}e^{at}$

⇓

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\}=\frac{t^{n-1}}{(n-1)!}\cdot e^{at}}$$

本題 $a=-1$, $n=3$.

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* $\mathcal{L}\left\{\frac{1}{s^2-4}\right\}=\mathcal{L}\frac{1}{2}\left\{\frac{2}{s^2-2^2}\right\}=\frac{1}{2}\sinh 2t$

* $\mathcal{L}\left\{\frac{1}{s^2-3}\right\}=\mathcal{L}\left\{\frac{1}{s^2-(\sqrt{3})^2}\right\}=\frac{1}{\sqrt{3}}\mathcal{L}\left\{\frac{\sqrt{3}}{s^2-(\sqrt{3})^2}\right\}=\frac{1}{\sqrt{3}}\sinh\sqrt{3}t$

逆転換重要公式： $s \longrightarrow t$

1. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = t\cdot e^{at}$

2. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)}\right\} = e^{at}$

3. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

4. $\mathcal{L}^{-1}\left\{\frac{2s}{(1+s^2)^2}\right\} = t\cdot \sin t$

5. $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

7. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

8. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

9. $\mathcal{L}^{-1}\left\{F(s-a)\right\} = f(t)\cdot e^{at}$. 平移定理.

重要