Differental Equations of Motion

Introduction : There are 6 unknowns for a genenal fluid mechanics problems <u>u . v . w . P . ρ . T</u>

Conservation of mass (1个方程式)

——————— momentum (3)

——————— energy (1)

Equation of state (1) [$PV = \rho RT$]
(狀態方程式)

§ Conservation of Mass

Derivation :

Lagrangian viewpoint

$$\frac{dm}{dt} = 0 \quad , \text{ mass is constant}$$

$$m = \rho \forall \quad , \quad \frac{d}{dt}(\rho \forall) = 0$$

$$\rho \frac{d\forall}{dt} + \forall \frac{d\rho}{dt} = 0 \quad , \quad 因 \ \frac{1}{\forall}\frac{d\forall}{dt} = \varepsilon_{ii} = \nabla \cdot \vec{V}$$ divided by $\rho\forall$.

$$\frac{1}{\forall}\frac{d\forall}{dt} + \frac{1}{\rho}\frac{d\rho}{dt} = 0$$

$$\Downarrow$$

$$\nabla \cdot \vec{V} + \frac{1}{\rho}\frac{d\rho}{dt} = 0 \quad , \quad \frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0$$

Recall substamtial derivative

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{V} \cdot \nabla)\rho \quad \text{代回}$$

$$\frac{\partial \rho}{\partial t} + \underline{(\vec{V} \cdot \nabla)\rho + \rho(\nabla \cdot \vec{V})} = 0$$

向量等式 $\nabla \cdot (a\vec{B}) = a(\nabla \cdot \vec{B}) + (\vec{B} \cdot \nabla) a$

$$\nabla \cdot (\rho \vec{V}) = \rho(\nabla \cdot \vec{V}) + (\vec{V} \cdot \nabla)\rho$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

continuity eqn. [質量守恆方程式]

1.1 Simplification for $\underline{\text{Incomp flow}}$ (ρ = constant)

Incompressiable flow

$\nabla \cdot \vec{V} = 0$, $\frac{\partial u_i}{\partial x_i} = 0$ even if flow is unsteady $\frac{\partial \rho}{\partial t} = 0$

1.2 Simplification for steady flow

$\frac{\partial}{\partial t} = 0$, $\nabla \cdot \rho \vec{V} = 0$, $\frac{\partial}{\partial x_j}(\rho u_j) = 0$

even for compressible flow

2. Stream Function ψ

※ Definition , Consider 2-D steady flow in x-y plane.

conti eqn. $\frac{\partial}{\partial x_j}(\rho u_j) = 0$

in 2-D

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0$$

Simplification of Energy Equation

$$\begin{cases} dh = C_p \, dT \\ de = C_v \, dT \end{cases} \qquad C_p, C_v = \text{Const}$$

$$\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + K \frac{\partial}{\partial x_i} \frac{\partial T}{\partial x_i} + \Phi$$

$$\underline{\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + K \nabla^2 T + \Phi}$$

valid for a perfect gas, not valid for an incompressible flow.

2) Incompressible Flow (and K = const)

$$\begin{cases} \text{perfect gas}, & P = \rho R T, & C_p = C_v + R \\ \text{incompressible}, & \rho = \text{const}, & C = C_p = C_v \end{cases}$$

$$\rho \frac{De}{Dt} + P \frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(K \frac{\partial T}{\partial x_i}\right) + \Phi$$

$$De = C \, DT$$

$$\rightarrow \quad \underline{\rho C_p \frac{DT}{Dt} = K \nabla^2 T + \Phi}$$

valid for incompressible flow only (not for perfect gas).

Define

$$\underline{\rho u = \frac{\partial \psi}{\partial y} \ , \quad -\rho v = \frac{\partial \psi}{\partial x}}$$

通式

代回原式

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Conti eqn. can be automatically satisfied.
→ one less eqn. , one less unknown.

Problem → eqns are one order higher

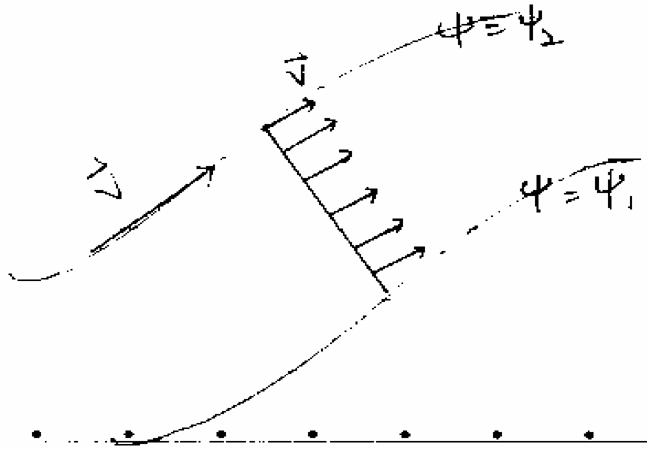
for incompressible flow.

Conti $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

define $\quad u = \frac{\partial \psi}{\partial y} \ , \quad v = -\frac{\partial \psi}{\partial x}$

Streamlines

ψ = constant along a streamline



between two streamlines

For compressible flow

$$\dot{m} = \int \rho u \, dy = \int_1^2 d\psi$$

$$\left(\rho u = \frac{\partial \psi}{\partial y}\right) = \psi_2 - \psi_1$$

For incompressible flow

$\dot{Q}$ (Flow rate) between two streamlines

$$= \int u \, dy = \int_1^2 d\psi = \psi_2 - \psi_1$$

§ Consentration of momentum

Newton's 2nd law $\vec{F} = m\vec{a}$

$$f = \frac{\vec{F}}{\forall} = \rho \vec{a} \quad (1)$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{D\vec{V}}{Dt} \quad (1) \text{ becomes}$$

$$\rho \frac{D\vec{V}}{Dt} = \vec{f} = \vec{f}_{body} + \vec{f}_{surf} \quad (2)$$

(a) body forces (per unit vol)

$$\vec{f}_{body} = \rho \vec{g}$$

(b) surface forces

$$\vec{f}_{surf} = \frac{\Sigma\,(\text{surface stresses})\,(\text{surface area})}{\text{volume}}$$

considerating fluid → particle, shear stress tensor, τ_{ij}.

= Stress in the j-direction, acting on a surface normal to the i-direction

From textbook, $f_{i_{surface}} = \dfrac{\partial \tau_{ij}}{\partial x_j}$. (2) becomes

$$\boxed{\begin{aligned} \rho \frac{D\vec{V}}{Dt} &= \rho\vec{g} + \nabla \cdot \tau_{ij} \\ \rho \frac{Du_i}{Dt} &= \rho g_i + \frac{\partial}{\partial x_j}\tau_{ij} \end{aligned}} \quad (3)$$

f_{body} + $f_{surface}$

We need to write τ_{ij} in terms of u_i, P, ρ, T.

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}(\tau_{ij}) \qquad (3)$$

Key : express τ_{ij} interms of u_i , P , ρ , T
(6 variables)

※ Inviscid flow

Since no friction , there can be no shear.

$$\tau_{ij} = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} = -P\delta_{ij}$$

$$\tau_{ij} = -P\delta_{ij}$$

(3) ⇒

$$\rho \frac{Du_i}{Dt} = \rho g_i \underbrace{- \frac{\partial}{\partial x_j}(P\delta_{ij})}_{-\frac{\partial P}{\partial x_i}}$$

$\delta_{ij} = 1$, if $i = j$

$\delta_{ij} = 0$, if $i \neq j$

So

$$\rho \frac{Du_i}{Dt} = -\frac{\partial P}{\partial x_i} + \rho g_i$$ Euler's eqn Vaid for <u>Inviscid flow.</u>

無剪應力,只有正向力

§ Newtonian Fluid
(Stoke's approch)

Stress is related to strain → $\tau_{ij} = \text{func}(\varepsilon_{ij}, P, T, \cdots)$

Stokes (1845) 應力和應變的關係，且應變可用速度表示

made 3 assumptions

(a) Fluid is continuum (連体) & Newtonian

* Stress = linear function for strain

τ_{ij} = linear function for ε_{ij}

(b) Fluid is isotropic (均方向性) 不隨座標改變

Stress - Strain relationship is independent of coordinate systems

(c) When $\varepsilon_{ij} = 0$ (no strain) — strain = 0

the only stress that exist are normal stress (pressure)

$$\tau_{ij} = -P\delta_{ij}, \quad \text{when } \varepsilon_{ij} = 0$$

，对沒應變，eqn 將回到 Euler's Eqn ~

Use assump (a) and (b)

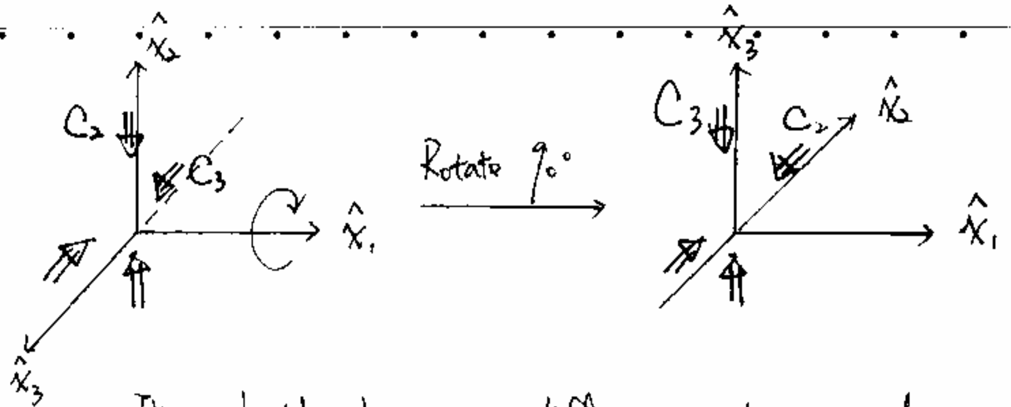
Let's pick the principal axes (使得 off-diagonal τ, ε 為 0)

Notation $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ = principal axes

$$\hat{\tau}_{ij} = \begin{pmatrix} \hat{\tau}_{11} & 0 & 0 \\ 0 & \hat{\tau}_{22} & 0 \\ 0 & 0 & \hat{\tau}_{33} \end{pmatrix}, \quad \hat{\varepsilon}_{ij} = \begin{pmatrix} \hat{\varepsilon}_{11} & 0 & 0 \\ 0 & \hat{\varepsilon}_{22} & 0 \\ 0 & 0 & \hat{\varepsilon}_{33} \end{pmatrix}$$

$$\hat{\tau}_{11} = C_1\hat{\varepsilon}_{11} + C_2\hat{\varepsilon}_{22} + C_3\hat{\varepsilon}_{33} \boxed{-P} \quad \leftarrow \text{Assumption C}$$

represent ε_{22} can influence τ_{11} with coefficient C_2



It should be no difference to coordinate change, thus → $C_2 = C_3$. 對 $\hat{\tau}_{11}$ 而言，$\hat{\varepsilon}_{22}$ $\hat{\varepsilon}_{33}$ 對其影响力相同.（均向性）

Similarly,

$$\hat{\tau}_{11} = C_1\hat{\varepsilon}_{11} + C_2\hat{\varepsilon}_{22} + C_2\hat{\varepsilon}_{33} - p + \boxed{C_2\hat{\varepsilon}_{11} - C_2\hat{\varepsilon}_{11}}$$

$$\hat{\tau}_{22} = C_1\hat{\varepsilon}_{22} + C_2\hat{\varepsilon}_{11} + C_2\hat{\varepsilon}_{33} - p + \boxed{C_2\hat{\varepsilon}_{22} - C_2\hat{\varepsilon}_{22}}$$

$$\hat{\tau}_{33} = C_1\hat{\varepsilon}_{33} + C_2\hat{\varepsilon}_{11} + C_2\hat{\varepsilon}_{33} - p + \boxed{C_2\hat{\varepsilon}_{33} - C_2\hat{\varepsilon}_{33}}$$

$$\hat{\tau}_{11} = -p + \underbrace{(C_1 - C_2)}_{K}\hat{\varepsilon}_{11} + \underbrace{C_2(\hat{\varepsilon}_{11} + \hat{\varepsilon}_{22} + \hat{\varepsilon}_{33})}_{\frac{\partial \hat{u}_1}{\partial \hat{x}_1} + \frac{\partial \hat{u}_2}{\partial \hat{x}_2} + \frac{\partial \hat{u}_3}{\partial \hat{x}_3} = \nabla\cdot\vec{V}}$$

整理後

$$\left.\begin{aligned}\hat{\tau}_{11} &= -p + K\hat{\varepsilon}_{11} + C_2\underline{\hat{u}_{k,k}}\\ \hat{\tau}_{22} &= -p + K\hat{\varepsilon}_{22} + C_2\underline{\hat{u}_{k,k}}\\ \hat{\tau}_{33} &= -p + K\hat{\varepsilon}_{33} + C_2\underline{\hat{u}_{k,k}}\end{aligned}\right\} \quad \hat{\tau}_{ij} = 0 \quad i \neq j$$

速度の散度

What's about the shear?

Rotate the axes three-dimensionally back to an arbitray axes. Use direction cosines.

再將 $\hat{\tau}_{11}$, $\hat{\tau}_{22}$, $\hat{\tau}_{33}$, $\hat{\varepsilon}_{11}$, $\hat{\varepsilon}_{22}$, $\hat{\varepsilon}_{33}$，用 τ_{11}, τ_{22}, τ_{33}, ε_{11}, ε_{22}, ε_{33} 替代。ex: $\tau_{11} = \hat{\tau}_{11} l_1^2 + \hat{\tau}_{22} m_1^2 + \hat{\tau}_{33} n_1^2$

$$\varepsilon_{11} = \hat{\varepsilon}_{11} l_1^2 + \hat{\varepsilon}_{22} m_1^2 + \hat{\varepsilon}_{33} n_1^2$$

$$\tau_{12} = \hat{\tau}_{11} l_1 l_2 + \hat{\tau}_{22} m_1 m_2 + \hat{\tau}_{33} n_1 n_2$$

$$\varepsilon_{12} = \hat{\varepsilon}_{11} l_1 l_2 + \hat{\varepsilon}_{22} m_1 m_2 + \hat{\varepsilon}_{33} n_1 n_2$$

x_1 對 $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ 之方向餘弦為 l_1, m_1, n_1

x_2 對 $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ 之方向餘弦為 l_2, m_2, n_2

$$l_1^2 + m_1^2 + n_2^2 = 1$$

get (此時不在主軸上)

$$\tau_{11} = -P + K\varepsilon_{11} + C_2 u_{k,k}$$

$$\tau_{22} = -P + K\varepsilon_{22} + C_2 u_{k,k}$$

$$\tau_{33} = -P + K\varepsilon_{33} + C_2 u_{k,k}$$

$$\left.\begin{aligned} \tau_{12} &= K\varepsilon_{12} \\ \tau_{23} &= K\varepsilon_{23} \\ \tau_{31} &= K\varepsilon_{31} \end{aligned}\right\} \text{另三對稱 } \tau_{21}, \tau_{32}, \tau_{13}$$

$K = 2\mu$, first coefficient of viscosity (一般的黏滯係数)

$C_2 = \lambda$, 2nd coefficient of viscosity or bulk viscosity (可压縮才会用到)

$$\Rightarrow \tau_{ij} = \underbrace{-P\delta_{ij}}_{\text{压力}} + \underbrace{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{剪力}} + \underbrace{\delta_{ij}\lambda\frac{\partial u_k}{\partial x_k}}_{\text{体積の拉伸}}$$

$$\mu, \lambda = \text{function of } T, P, \rho$$

§ Newtonian Fluid

a) Total stress tensor (Including pressure term)

$$\tau_{ij} = -P\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + \delta_{ij}\lambda u_{k,k}$$

b) Stress tensor without pressure term.

$$\tau'_{ij} = \mu(u_{i,j}+u_{j,i}) + \delta_{ij}\lambda \cancel{u_{k,k}}^{\,0}$$

deviatoric stress tensor (Viscous)

Coefficient of Bulk viscosity, λ

(a) For incompressible flows, $\cancel{\nabla\cdot\vec{V} = u_{k,k} = 0}$

So λ never gets into the problem.

(b) For compressible flows, $\lambda \approx \frac{2}{3}\mu$ for most fluids

§ Navier - Stokes Equations

Substitute our τ_{ij} into our momentum eqn

(3) 慣性力 $\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}$ → 代 前面所推導の公式

$\rho\vec{a}$ = 合下

$$\Rightarrow \rho\frac{Du_i}{Dt} = \rho g_i + \underbrace{\frac{\partial}{\partial x_j}(-P\delta_{ij})}_{-\frac{\partial P}{\partial x_i}} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right) + \underbrace{\frac{\partial}{\partial x_j}(\delta_{ij}\lambda u_{k,k})}_{\frac{\partial}{\partial x_i}(\lambda u_{k,k})}$$

Newtonian Isotropic Compressible fluid flow

※ Simplification, Incompressible Flows

continuity → $\nabla\cdot\vec{V} = 0$ ($u_{k,k} = 0$)

ρ = const
T = const } implies $\mu, \lambda \approx$ const, k (thermal conductivity) ≈ Const

$$\rho\frac{Du_i}{Dt} = \rho g_i - \frac{\partial P}{\partial x_i} + \mu u_{i,jj} \leadsto \nabla^2\vec{V}$$

$$\rho\frac{Du_i}{Dt} = \rho\vec{g} - \nabla P + \mu\nabla^2\vec{V}$$

重力 壓力 黏滯力 1845 → 2007

§ Conservation of Energy

$$\frac{DE_t}{Dt} = \frac{DQ}{Dt} + \frac{DW}{Dt}$$

- Combine $\frac{DE_t}{Dt}$, $\frac{DQ}{Dt}$, $\frac{DW}{Dt}$ into (2.33)

$\div \forall$

$$\rightarrow \frac{1}{\forall}\frac{DE_t}{Dt} = \frac{1}{\forall}\frac{DQ}{Dt} + \frac{1}{\forall}\frac{DW}{Dt}$$

(Eq. 2.34) (Eq. 2.35) (Eq. 2.37)

- $$\rho\frac{De}{Dt} + \boxed{\underline{\underline{\rho u_i \frac{Du_i}{Dt}}} - \underline{\underline{\rho g u_i}}} = \frac{\partial}{\partial x_i}\left(K\frac{\partial T}{\partial x_i}\right) + \underline{\frac{\partial}{\partial x_j}(u_i \tau_{ij})} \quad (2)$$

$\Downarrow$ From product rule

$$\frac{\partial}{\partial x_j}(u_i\tau_{ij}) = u_i\frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij}\frac{\partial u_i}{\partial x_j} \quad (3)$$

- momentum eqn.

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}$$

$$\boxed{\rho u_i\frac{Du_i}{Dt} = \rho u_i g_i} + u_i\frac{\partial}{\partial x_j}\tau_{ij}$$

- So $\frac{\partial}{\partial x_j}(u_i\tau_{ij}) = \rho u_i\frac{Du_i}{Dt} - \rho u_i g_i + \tau_{ij}\frac{\partial u_i}{\partial x_j}$ 代入 (2)

$$\rightarrow \boxed{\rho\frac{De}{Dt} = \frac{\partial}{\partial x_i}\left(K\frac{\partial T}{\partial x_i}\right) + \tau_{ij}\frac{\partial u_i}{\partial x_j}}$$

$$\tau_{ij} = \tau_{ij}' - p\delta_{ij}$$ (將壓力項提出)

- $$\rho\frac{De}{Dt} + p\delta_{ij}\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(K\frac{\partial T}{\partial x_i}\right) + \underbrace{\tau_{ij}'\frac{\partial u_i}{\partial x_j}}$$

Φ = dissipation function
消散函數

又 $h = e + \frac{p}{\rho}$ — enthalpy (焓)

$$\rightarrow \rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \underbrace{\frac{\partial}{\partial x_i}\left(K\frac{\partial T}{\partial x_i}\right)} + \Phi$$

$h = h(p, T)$

$\nabla\cdot(K\nabla T)$

Simplification of Energy Equation

$dh = C_p\, dT$

$de = C_v\, dT \qquad C_p,\ C_v = \text{Const}$

$$\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + K \frac{\partial}{\partial x_i}\frac{\partial T}{\partial x_i} + \Phi$$

$$\underline{\rho C_p \frac{DT}{Dt} = \frac{DP}{Dt} + K \nabla^2 T + \Phi}$$

valid for a perfect gas, not valid for an incompressible flow.

2) Incompressible Flow (and $K = \text{const}$)

$\begin{cases} \text{perfect gas}, & P = \rho R T, \quad C_p = C_v + R \\ \text{incompr.}, & \rho = \text{const}, \quad C = C_p = C_v \end{cases}$

$$\rho \frac{De}{Dt} + P \frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(K \frac{\partial T}{\partial x_i}\right) + \Phi$$

$$De = C\, DT$$

$\rightarrow \underline{\rho C_p \frac{DT}{Dt} = K \nabla^2 T + \Phi}$ valid for incompressible flow only (not for perfect gas).

◎ Boundary Conditions

→ Fluid - Solid Interface

no slip condition $\quad \vec{V}_f = \vec{V}_{wall}$ at wall

$T_f = T_{wall}$ or $q_{wall} = -k \frac{\partial T}{\partial n}\Big|_{wall}$

絕緣

For insulated wall → $q_w = 0$, $\frac{\partial T}{\partial n} = 0$

n : 壁面內

◎ Initial Conditions

if unsteady , $\vec{V}$ 、P 、T 、ρ everywhere at $t=0$ are required to be specified

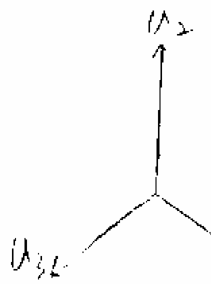
◎ Control surfaces

Eqns all elliptic specify $\vec{V}$ 、P 、T 、ρ at all points on control surface for all time.

正交

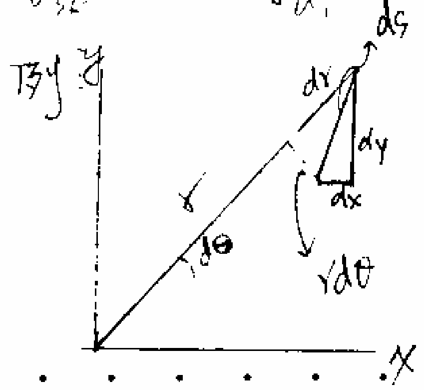
Orthogonal Coordinate Systems (除了原点, 其餘軸與軸皆互垂直

Orthogonal : coordinate axes are everywhere ⊥ to each other . (Except possibly for a singularity usually at origin)



$$ds^2 = (h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2$$

例子



$$ds^2 = dx^2 + dy^2 = dr^2 + (r d\theta)^2$$

for $u_1 = r$, $u_2 = \theta$

$h_1 = 1$, $h_2 = r$

Vorticity

$$\vec{\omega} = \nabla \times \vec{V}$$

Navier - Stokes eqn in terms of vorticity

Assumptions : $\begin{cases} 1. & \text{constant } \mu \\ 2. & \text{constant } \rho \\ 3. & \vec{g} = -g\vec{k} \end{cases}$

Continuity :

$$\nabla \cdot \vec{V} = 0$$

Momentum :

$$\rho \frac{D\vec{V}}{Dt} = \rho \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right) = -\nabla p - \rho g \vec{k} + \mu \nabla^2 \vec{V}$$

Vector identity

$$(\vec{V} \cdot \nabla)\vec{V} \equiv \nabla \left(\frac{|\vec{V}|^2}{2} \right) - \vec{V} \times \underbrace{(\nabla \times \vec{V})}_{\vec{\omega}}$$

$|\vec{V}|$ = magnitude of $|\vec{V}|$

$$\nabla^2 \vec{V} \equiv \nabla(\nabla \cdot \vec{V}) - \nabla \times \underbrace{(\nabla \times \vec{V})}_{\vec{\omega}}$$

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \nabla \left(\frac{|\vec{V}|^2}{2} \right) - \vec{V} \times \vec{\omega} \right) = -\nabla p \underline{- \rho g \vec{k}} - \mu \nabla \times \vec{\omega}$$

$$\nabla \rho g z \rightarrow \nabla z = \cancelto{0}{\frac{\partial z}{\partial x}} \vec{i} + \cancelto{0}{\frac{\partial z}{\partial y}} \vec{j} + \cancelto{1}{\frac{\partial z}{\partial z}} \vec{k}$$

$$\Rightarrow \quad \rho \frac{\partial \vec{V}}{\partial t} + \nabla \left(p + \frac{1}{2} \rho |\vec{V}|^2 + \rho g z \right) = \rho \vec{V} \times \vec{\omega} - \mu \nabla \times \vec{\omega}$$

valid for unsteady, viscous, rotational, incompressible.

◎ unsteady, incompressible, irrotational ($\vec{\omega}=0$), viscous flows

$\rightarrow$ if $\nabla \times \vec{V} = \vec{\omega} = 0$

then $\vec{V} = \nabla \phi$ (ϕ = velocity potentional)

$\nabla \times (\nabla \phi) \equiv 0$ (等式)

$$\Rightarrow \quad \nabla\left(\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho|\vec{V}|^2 + \rho g z\right) = 0$$

$(\quad) =$ Constant everywhere

◎ Unsteady Bernoulli Eqn.

Valid even for viscous flow if it is irrotational

◎ Steady incompressible, irrotational viscous

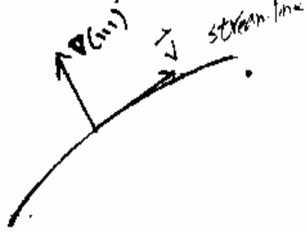
$$\underline{p + \frac{1}{2}\rho|\vec{V}|^2 + \rho g z = \text{Constant everywhere}}$$

Steady Bernoulli Eqn.

3) Steady, rotational but inviscid trick.

$$\vec{V}\cdot \text{Eq (2)}$$

$$\vec{V}\cdot\nabla\left(p+\frac{1}{2}\rho|\vec{V}|^2+\rho g z\right)=\rho\overbrace{\vec{V}\cdot\underbrace{(\vec{V}\times\vec{\omega})}_{\vec{V}\times\vec{\omega}\text{ 產生の向量垂直於 }\vec{V}\text{ 及 }\vec{\omega}}}^{0}-\underbrace{\mu}_{0\,(\text{inviscid})}(\vec{V}\cdot\nabla\times\vec{\omega})$$



$$\vec{V}\cdot\nabla(\cdots)=0$$

imply $\vec{V}\perp\nabla(\cdots)$

$\Rightarrow\ p+\frac{1}{2}\rho|\vec{V}|^2+\rho g z=\text{Const}$ (along a streamline)

2. Vorticity Eqn.

Incomp w/ const props

Start with N-S eqn for incomp flow

$$\rho\frac{\partial\vec{V}}{\partial t}+\rho(\vec{V}\cdot\nabla)\vec{V}=-\nabla p-\rho g\vec{k}+\mu\nabla^2\vec{V}\quad(1)$$

identity $\quad(\vec{V}\cdot\nabla)\vec{V}=\nabla\left(\frac{|\vec{V}|^2}{2}\right)-\vec{V}\times\vec{\omega}$

$$\frac{\partial\vec{V}}{\partial t}+\nabla\left(\frac{1}{2}|\vec{V}|^2+\frac{p}{\rho}+gz\right)=\vec{V}\times\vec{\omega}+\nu\nabla^2\vec{V}\quad(2)$$

Take curl of (2) i.e. $\nabla\times[2]$

$$\frac{\partial\vec{\omega}}{\partial t}+\nabla\times\nabla(\cdots)=\nabla\times(\vec{V}\times\vec{\omega})+\underline{\nu\nabla\times\nabla^2\vec{V}}\quad \nu\nabla^2\underbrace{(\nabla\times\vec{V})}_{\vec{\omega}}$$

$$\rightarrow -\vec{V}\cdot\nabla\vec{\omega}+(\vec{\omega}\cdot\nabla)\vec{V}$$

identy

(1) $\nabla\times\nabla A\equiv 0$ for scalar A

(2) $\nabla\times\nabla^2\vec{V}\equiv\nabla^2(\nabla\times\vec{V})=\nabla^2\vec{\omega}$

(3) $\nabla\times(\vec{V}\times\vec{\omega})\equiv-(\vec{V}\cdot\nabla)\vec{\omega}+(\vec{\omega}\cdot\nabla)\vec{V}+\vec{V}\overbrace{(\cancel{\nabla\cdot\vec{\omega}})}^{0}-\vec{\omega}\overbrace{(\cancel{\nabla\cdot\vec{V}})}^{0}$

Finally

$$\underbrace{\frac{D\vec{\omega}}{Dt}}_{\left(\frac{\partial\vec{\omega}}{\partial t}+\vec{V}\cdot\nabla\vec{\omega}\right)}=\overbrace{(\vec{\omega}\cdot\nabla)\vec{V}}^{\text{vortex streching}}+\nu\nabla^2\vec{\omega}$$

Convection of vorticity ($\frac{D\vec{\omega}}{Dt}$)

Viscous Diffusion of vorticity ($\nu\nabla^2\vec{\omega}$)

2.1 Sumplification of 2-D flow

in (X-Y) plane

$\vec{w} = (0, 0, w_z) \rightarrow w_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad W \approx 0, \frac{\partial}{\partial z} \approx 0$

z - component of (4) $\left(\frac{\partial}{\partial z} \approx 0\right)$

$$\frac{\partial w_z}{\partial t} + u\frac{\partial w_z}{\partial x} + v\frac{\partial w_z}{\partial y} + \cancel{W}^{0}\frac{\partial w_z}{\partial z} = (\vec{w}\cdot\nabla)\cancel{W}^{0} + \nu\left(\frac{\partial^2 w_z}{\partial x^2} + \frac{\partial^2 w_z}{\partial y^2} + \cancel{\frac{\partial^2 w_z}{\partial z^2}}^{0}\right) \quad (4)$$

Also for 2-D incomp flow

$$u = \frac{\partial \psi}{\partial y}, \quad v = \frac{-\partial \psi}{\partial x}, \quad w_z = -\nabla^2\psi$$

$\nabla^2\psi = -w_z$, if irrot 2-D flow

$\nabla^2\psi = 0$

then

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^4\psi$$

Vorticity eqn in terms of ψ ∴ 只解 ψ (流線函数)

1 eqn, 1 unknown → but 4th order

2.2 For low Re flows

2-D Incomp $Re \ll 1$ → Stokes flow or Creeping flow

$Re = \dfrac{\text{Interial terms}}{\text{Viscous terms}}$, If $Re \ll 1$

$\Rightarrow \nabla^4\psi = 0$ Biharmonic eqn.