III. Laminar Solutions of N-S eqn.

◎ Consider Incomp flow, w/ const props & small ΔT

5 eqns, 5 unknowns, u, v, w, p, T

Conti $\nabla \cdot \vec{V} = 0$ (1 eqn)

mom $\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V}$ (3 eqn)

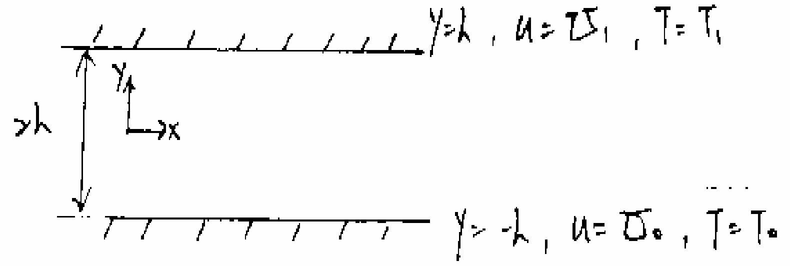
Energy $\rho C_p \frac{DT}{Dt} = k \nabla^2 T + \Phi$ (1 eqn)

energy eqn is uncoupled from conti & momentum eqn.

Example of Exact Laminar Solutions

◎ Flow between infinite // plates } parallel

平行板



Assumptions

(1) Incompressible (small ΔT)

const properties

(2) 2-D flow (no flow into or out of x-y plane)

$\left(w = 0, \ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial T}{\partial z} = 0 \right)$

(3) // flow, $(v = 0)$

(4) gravity neglect in x-y plane

Eqns of Motion

Conti $\quad \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{0} + \cancel{\frac{\partial w}{\partial z}}^{0} = 0 \qquad (\nabla \cdot \vec{V}) = 0$

$\qquad\qquad\qquad\qquad (3) \qquad (2)$

$\frac{\partial u}{\partial x} = 0 \qquad u$ is not a function of x & z

$u = u(y, t)$

x- mom :

$$\rho \left\{ \underbrace{\frac{\partial u}{\partial t}}_{} + \underbrace{u\cancel{\frac{\partial u}{\partial x}}}_{\text{Conti}} + \underbrace{v\cancel{\frac{\partial u}{\partial y}}}_{(3)} + \underbrace{w\cancel{\frac{\partial u}{\partial z}}}_{(2)} \right\} = \underbrace{\rho \cancel{g_x}}_{(4)} - \frac{\partial p}{\partial x} + \mu \left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{Conti}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{(2)} \right)$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2}$$

y -mom : $\quad \rho \underbrace{\cancel{\left\{ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + \cdots \right\}}}_{(3)} = \underbrace{\rho \cancel{g_y}}_{(4)} - \frac{\partial p}{\partial y} + \mu \underbrace{(\nabla^2 \cancel{v})}_{(3)}$

$$\Rightarrow \frac{\partial p}{\partial y} = 0 \qquad p \text{ is not a function of } y.$$

z - mom : $\quad \rho \underbrace{\cancel{[\ \cdots\ ]}}_{(2)} = \rho g_z - \frac{\partial p}{\partial z} + \mu \underbrace{\cancel{(\nabla^2 w)}}_{(2)}$

$\frac{\partial p}{\partial z} = \rho g_z \quad , \quad \vec{g} = -g\vec{k} \quad , \quad \frac{\partial p}{\partial z} = -\rho g$

$p = -\rho g_z + f(x, t)$

hydrostatic pressure

define

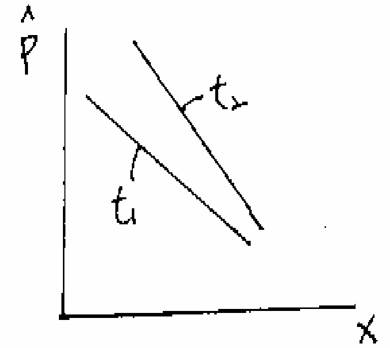
$\hat{p} = p + \rho g_z \quad , \quad \hat{p} = f(x, t)$

So

$$\frac{\partial}{\partial x}\left\{ \rho \frac{\partial u}{\partial t} = -\frac{\partial \hat{p}}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \right\}$$

$$\rho \cancel{\frac{\partial^2 u}{\partial x \partial t}}^{\,0} = -\frac{\partial}{\partial x}\left(\frac{\partial \hat{p}}{\partial x}\right) + \mu \cancel{\frac{\partial^3 u}{\partial x \partial y^2}}^{\,0} \rightarrow \left(\frac{\partial u}{\partial x}\right)$$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \hat{p}}{\partial x}\right) = 0$ $\quad \hat{p}$ is a linear function of x.



Energy eqn.

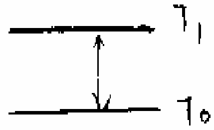
$T = T_1$ entire plate $\Rightarrow T \neq T(x)$

$$\rho C_p \left[\frac{\partial T}{\partial t} + \cancel{u \frac{\partial T}{\partial x}} + \underset{(3)}{\cancel{v \frac{\partial T}{\partial y}}} + \underset{(2)}{\cancel{w \frac{\partial T}{\partial z}}} \right] = K\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} + \underset{(2)}{\cancel{\frac{\partial^2 T}{\partial z^2}}} \right) + \Phi$$

$$\Phi = \mu\{[\sim]\} + \lambda (\cancel{\nabla \cdot \vec{V}})^0$$

$$= \mu \left(\frac{\partial u}{\partial y}\right)^2 \longrightarrow \text{速度的微分}$$

$$\Rightarrow \rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$



為非線性

$\frac{\partial T}{\partial y} = C$

$T = C_1 y + C_2$

為線性[illegible]

[illegible]

[illegible]項[illegible]熱源

Steady flow sol.

$$\left.\begin{array}{ll} u = u(y) & \text{only} \\ p = p(x) & \text{only} \\ T = T(y) & \text{only} \end{array}\right\} \quad \frac{dp}{dx} = \text{Const}$$

integrate,

x-mom : $\dfrac{d^2u}{dy^2} = \dfrac{1}{\mu}\dfrac{dp}{dx} \rightarrow$ O.D.E

Energy : $0 = k\dfrac{d^2T}{dy^2} + \mu\left(\dfrac{du}{dy}\right)^2$

Assume B.C. $U_1 = U$, $y = h$ for all time

$U_2 = 0$, $y = -h$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

$$\xrightarrow{\text{Int}} \quad u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)y^2 + C_1 y + C_2$$

Use B.Cs to find constants

at $y = -h$, $u = 0 \rightarrow 0 = \dfrac{h^2}{2\mu}\left(\dfrac{dp}{dx}\right) - C_1 h + C_2$

$y = h$, $u = U$, $\rightarrow U = \dfrac{h^2}{2\mu}\left(\dfrac{dp}{dx}\right) + C_1 h + C_2$

$$C_1 = \frac{U}{2h}, \quad C_2 = \frac{U}{2} - \frac{h^2}{2\mu}\left(\frac{dp}{dx}\right)$$

$$u(y) = \frac{y^2 - h^2}{2\mu}\left(\frac{dp}{dx}\right) + \frac{U}{2h}y + \frac{U}{2}$$

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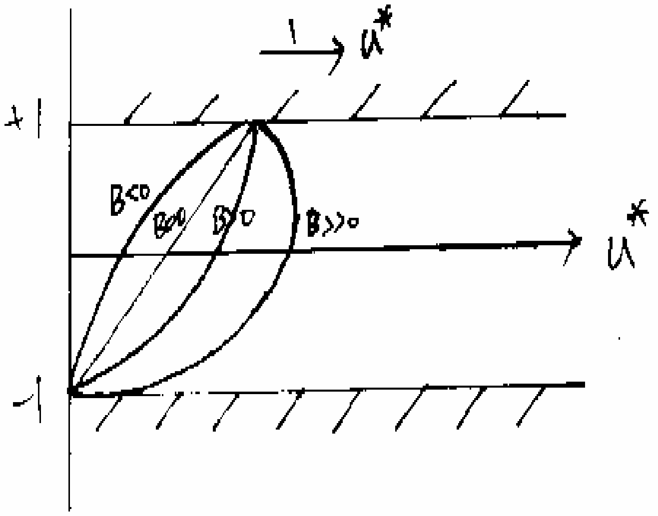
Non-dimensonal form

$u^* = \frac{u}{U}$, $y^* = \frac{y}{h}$

B= pressure gradient parameter $= -\frac{h^2}{\mu U}\left(\frac{dp}{dx}\right)$

$$u^* = \frac{(1-y^{*2})}{2}B + \frac{y^*+1}{2}$$

General steady flow sol. for flow between 2 infinite // plates.



$y^* = 0 \rightarrow u^* = \frac{B}{2} + \frac{1}{2}$

$B = 0$, $u^* = \frac{1}{2}$

$B > 0$, $u^* > \frac{1}{2}$

$B < 0$, $u^* < \frac{1}{2}$

* $B >> 0$, u^* will > 1

若不可壓縮流場
中，P是 Const 時，
溫度可由速度來推.

Energy Equation.

$$K \frac{d^2T}{dy^2} = -\mu \left(\frac{du}{dy}\right)^2$$

non - dimensionalize

$$\begin{cases} y^* = y/h \\ u^* = u/U_s \end{cases}$$

define $\boxed{T^* = \frac{T - T_0}{T_1 - T_0}}$

at $y = h$, $T = T_1$, $T^* = 1$

$y = -h$, $T = T_0$, $T^* = 0$

$$\frac{dT}{dy} = \frac{dT}{dT^*} \frac{dT^*}{dy^*} \frac{dy^*}{dy} \quad , \quad T = T_0 + T^*(T_1 - T_0)$$

$$\frac{d[T_0 + T^*(T_1 - T_0)]}{dT^*} = (T_1 - T_0) \qquad \frac{dy^*}{dy} = \frac{d(\frac{y}{h})}{dy} = \frac{1}{h}$$

$$\frac{dT}{dy} = \frac{dT}{dT^*} \frac{dT^*}{dy^*} \frac{dy^*}{dy} = (T_1 - T_0)\left(\frac{dT^*}{dy^*}\right)\frac{1}{h}$$

$$\frac{d^2T}{dy^2} = \frac{d}{dy^*}\frac{dy^*}{dy}\left(\frac{dT}{dy}\right) = \frac{T_1 - T_0}{h^2} \frac{d^2T^*}{dy^{*2}}$$

$$\frac{du}{dy} = \frac{U_s}{h} \frac{du^*}{dy^*}$$

原 $K\frac{d^2T}{dy^2} = -\mu\left(\frac{du}{dy}\right)^2$

$$\frac{d^2T^*}{dy^{*2}} = \frac{-\mu}{K} \frac{h^2}{(T_1 - T_0)} \frac{U_s^2}{h^2} \left(\frac{du^*}{dy^*}\right)^2$$

$$\frac{du^*}{dy^*} = \frac{1}{2} - By^*$$

Let Br = Brinkman # $= \dfrac{\mu U^2}{k(T_1 - T_0)}$

$$\frac{d^2T^*}{dy^{*2}} = -Br\left(\frac{1}{4} - By^* + B^2 y^{*2}\right)$$

Integrate twice:

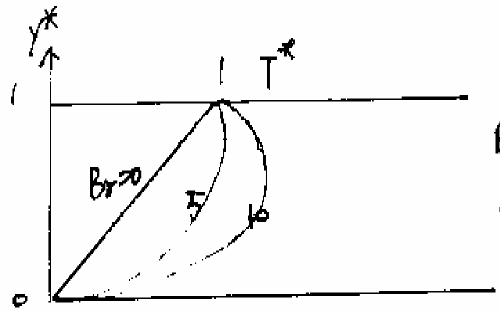
$$T^* = \frac{1}{2}(1+y^*) + \frac{Br}{8}(1-y^*)^2 - \frac{B\,Br}{6}(y^* - y^{*3}) + \frac{B^2 Br}{12}(1-y^{*4})$$

case a). $B=0$ (No pressure gradient → pure couette flow)

/Kwet/

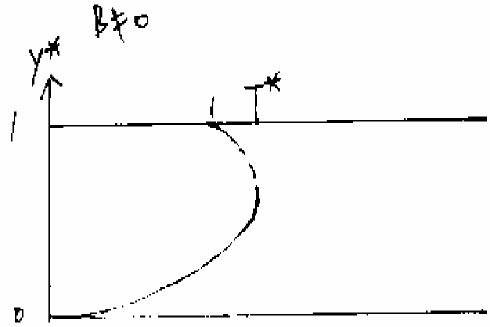
only valid for small ΔT

→ assume constant properties



$Br=0 \rightarrow U=0$, no flow fluid -- conduction.

case b). $B \neq 0$



H.W -2

prob 2.18

prob 3.8

(air at 20°C

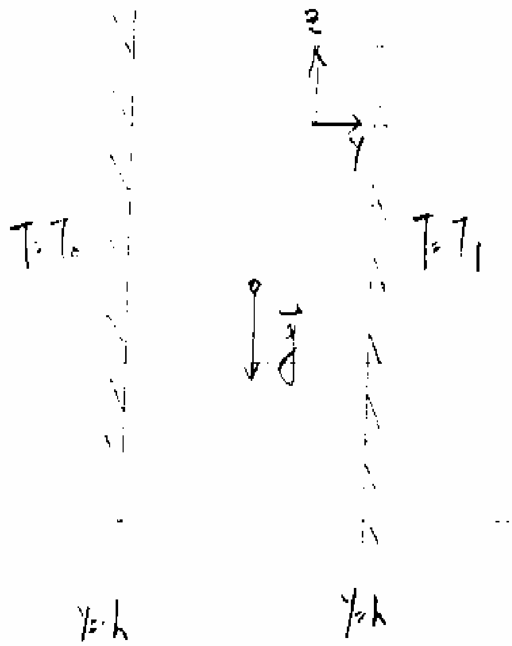
$\mu = 1.81 \times 10^{-5}$ Pa-s)

1.2 Non-Newtonian Couette Flow

e.g. Suppose

$$\tau_{ij} = -p\,\delta_{ij} + \underset{\substack{\downarrow \\ \text{const}}}{a}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^{\boxed{b}}$$

2) Free Convection between Vertical plates



Assume:

1) zero-pressure gradient
2) Infinite, stationary walls
3) Steady flow $\left(\frac{\partial}{\partial t} = 0\right)$
4) 2-D $\left(u = \frac{\partial}{\partial x} = 0\right)$
5) // flow $(v = 0)$
6) ρ is a weak function of T (ΔT is small)
   (slightly compressible) (稍可压缩)
7) μ & k are constants

If without flow $\frac{\partial p}{\partial z} = -\rho g$

with flow

$$\rho \frac{DW}{Dt} = -\frac{\partial p}{\partial z} - \rho g + \mu \nabla^2 W$$

In general $\rho = \rho(p, T)$

$$d\rho = \cancel{\left.\frac{\partial \rho}{\partial p}\right)_T dp} + \left.\frac{\partial \rho}{\partial T}\right)_p dT \quad - A'$$

Assume ρ = function of T but not p

$\beta \equiv$ coefficient of thermal expansion

$$\equiv -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right)_p$$

$$d\rho = -\rho\beta\, dT$$
$$= -\rho_0 \beta\, dT$$

let $\rho = \rho_0 + d\rho$

$$\rho = \rho_0 - \rho_0\beta(T - T_0)$$

"slightly compressible flow assumption"

@ assume $\rho = \rho_0$ = constant for all terms, except the gravity term.

z-mom

$$\rho_0 \frac{Dw}{Dt} = \underline{-\frac{\partial p}{\partial z} - \rho_0 g} + \overbrace{\rho_0 \beta (T - T_0) g}^{\text{Buoyancy term 浮力項}} + \mu \nabla^2 w$$

let $\hat{p} = p + \rho_0 g z$

= hydrodynamic pressure + total hydrostatic pressure
動壓力　靜壓力

zero pressure gradient

$$\rightarrow \frac{\partial \hat{p}}{\partial z} = 0$$

z-mom

$$\underbrace{\rho_0 \frac{Dw}{Dt}}_{\text{慣性力}} = -\cancel{\frac{\partial \hat{p}}{\partial z}}^{\,0} + \underbrace{\rho_0 \beta (T - T_0) g}_{\text{浮力}} + \underbrace{\mu \nabla^2 w}_{\text{黏滯力}}$$

conti

$$\underset{3)}{\cancel{\frac{\partial \rho}{\partial t}}} + \underset{4)}{\cancel{\frac{\partial}{\partial x}\rho u}} + \underset{5)}{\cancel{\frac{\partial}{\partial y}\rho v}} + \frac{\partial}{\partial z}\rho w = 0$$

ρ_0

$$\rho \frac{\partial w}{\partial z} + w \cancel{\frac{\partial \rho}{\partial z}} = 0$$

Since wall temp is constant, $T \neq f(z)$

$\rightarrow T = T(y)$ only

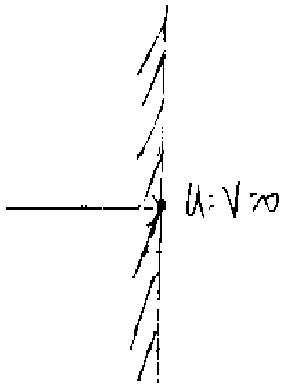
$\rightarrow \rho = \rho(y)$ only

$$\frac{\partial w}{\partial z} = 0, \quad w = w(y) \text{ only}$$

y-mom

$$\rho \cancel{\frac{Dv}{Dt}} = -\frac{\partial p}{\partial y} + \cancel{\rho g_y} + \mu \nabla^2 \cancel{v}$$

5) $\qquad$ z dir only $\qquad$ 5)



$$\Rightarrow \frac{\partial p}{\partial y} = 0 \;, \quad \frac{\partial p}{\partial x} = 0 \;4) \;, \quad \frac{\partial p}{\partial t} = 0 \;3)$$

$p = p(z)$ only , but $\hat{p} = \hat{p}(z)$

So $\hat{p}$ = const everywhere (在 free convection 中，以不考慮 $\hat{p}$，以 $\hat{p}$ = const)

z-mom

$$\rho_0 \left(\cancel{\frac{\partial w}{\partial t}} + u \cancel{\frac{\partial w}{\partial x}} + v \cancel{\frac{\partial w}{\partial y}} + w \cancel{\frac{\partial w}{\partial z}} \right) = -\cancel{\frac{\partial \hat{p}}{\partial z}}^{\,1)} + \rho_0 \beta (T - T_0) g$$

3) $\qquad$ 4) $\qquad$ 5) $\qquad$ conti

$$+\mu \left(\cancel{\frac{\partial^2 w}{\partial x^2}} + \frac{\partial^2 w}{\partial y^2} + \cancel{\frac{\partial^2 w}{\partial z^2}} \right)$$

4) $\qquad$ conti

$$\Rightarrow \mu \frac{d^2 w}{dy^2} = -\rho_0 \beta g (T - T_0)$$

Energy

$$\rho_0 C_p \frac{DT}{Dt} = K \nabla^2 T + \Phi$$

$$0 = K \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{dw}{dy} \right)^2$$

$$Gr = \text{Grashof } \# = \frac{g \rho^2 h^3 \beta (T - T_m)}{\mu^2}$$

$$J = \text{nameless} = \frac{\beta g \mu h}{k}, \quad J \ll 1 \text{ for all fluids}$$

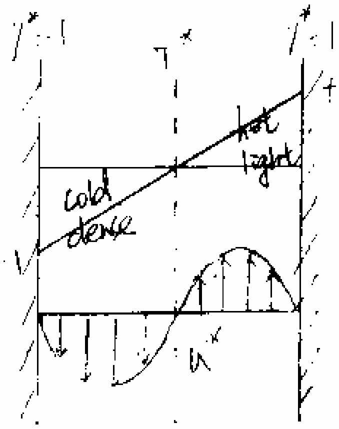
$$\frac{d^2 T^*}{dy^{*2}} = -\frac{J}{Gr}\left(\frac{dw^*}{dy^*}\right)^2 \qquad J \ll Gr$$

So $$\frac{d^2 T^*}{dy^{*2}} = 0 \quad \rightarrow \quad \text{sol'n} \quad T^* = y^*$$

mom eqn. $$\frac{d^2 w^*}{dy^{*2}} = -Gr\, y^*$$

B.C. $w^* = 0$, @ $y^* = \pm 1$

$$w^* = \frac{Gr}{6}(y^* - y^{*3})$$



define 寬長比為10倍
為無限長板

3. Unsteady // flow

3.1 Rayleigh Problem Impulse start of an infinite flat plate.