1.3 可分離常微分方程(separable ordinary differential equations，ODE)

很多一階常微分方程可簡化為

$$g(y)y' = f(x) \qquad \Rightarrow \qquad g(y)dy = f(x)dx$$

兩邊對 x 積分可得到其解為

$$\int g(y)\, dy = \int f(x)\, dx + c.$$

Example 1.　ODE　$y' = 1 + y^2$

原式為　$\dfrac{dy}{dx} = 1 + y^2 \qquad \Rightarrow \qquad \dfrac{dy}{1+y^2} = dx$

兩邊對 x 積分可得　$\tan^{-1} y = x + c \qquad \Rightarrow \qquad y = \tan(x+c)$

注意：$y = \tan x + c$　不是 ODE　$y' = 1 + y^2$　的解答

Example 2.　ODE　$y' = -2xy$,　　$y(0) = 1$

原式為　$\dfrac{dy}{dx} = -2xy \qquad \Rightarrow \qquad \dfrac{dy}{y} = -2xdx$

兩邊對 x 積分可得　$\ln|y| = -x^2 + \tilde{c} \qquad \Rightarrow \qquad y = e^{-x^2+\tilde{c}} = e^{\tilde{c}} \cdot e^{-x^2} = c \cdot e^{-x^2}$

從起始值　$y(0) = c \cdot e^0 = c = 1 \qquad \Rightarrow \qquad c = 1$　　所以特解為　$y = e^{-x^2}$

□ 可簡化為可分離形式(reduction to separable form)

ODE　$y' = f\left(\dfrac{y}{x}\right)$

其中 f 是任意的可微分的 y/x 函數，例如為 $\sin(y/x), (y/x)^4$ 等等，此 ODE 亦可稱為均勻(homogeneous)ODE。

讓　$y/x = u \qquad \Rightarrow \qquad y = ux$　微分得到　$y' = u'x + u = f(u)$

$\therefore\ u'x = f(u) - u \qquad \Rightarrow \qquad \dfrac{du}{f(u)-u} = \dfrac{dx}{x}$