1.4 正合常微分方程式和積分因子(exact ODE, integration factors)

若一函數 $u(x,y)$ 具有連續偏微分導數，則其**全微分**(total differential)爲

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

由上式可知若 $u(x,y)=c=$ 常數，則 $du=0$

Ex. 若 $u=x+x^2y^3=c$ ，則

$$du=dx+2xy^3dx+3x^2y^2dy=0 \quad 或$$

$$y'=-\frac{1+2xy^3}{3x^2y^2} \quad \Rightarrow 也就是\ u=x+x^2y^3=c\ 是此\ ODE\ 的解。$$

◎ 方程式 $M(x,y)dx+N(x,y)dy=0$ 是屬於正合(exact)一階常微分方程式(正合條件爲 $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$)，若且爲若 (if and only if) 在整個所定義區域內有一函數 $u(x,y)$ 使得 $M=\frac{\partial u}{\partial x}$ 和 $N=\frac{\partial u}{\partial y}$ 成立。

將 $M=\frac{\partial u}{\partial x}$ 和 $N=\frac{\partial u}{\partial y}$ 代入 $M(x,y)dx+N(x,y)dy=0$ ，得到

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0 \quad \Rightarrow \quad u(x,y)=c=\ 常數是此\ ODE\ 的解$$

對 $M=\frac{\partial u}{\partial x}$ 積分，可得到 $\Rightarrow$ $u(x,y)=\int M(x,y)dx+k(y)$ ，或

對 $N=\frac{\partial u}{\partial y}$ 積分，可得到 $\Rightarrow$ $u(x,y)=\int N(x,y)dx+l(y)$

Example 1. $\cos(x+y)dx+(3y^2+2y+\cos(x+y))dy=0$

Sol. $M=\cos(x+y)$, $N=3y^2+2y+\cos(x+y)$

$$\frac{\partial M}{\partial y}=-\sin(x+y), \qquad \frac{\partial N}{\partial x}=-\sin(x+y) \qquad \Rightarrow \quad 正合$$

所以 $u=\int Mdx+k(y)=\int\cos(x+y)dx+k(y)=\sin(x+y)+k(y)$

爲了要得到 $k(y)$ 將上式對 y 微分

$$\frac{\partial u}{\partial y}=\cos(x+y)+k'(y)\equiv N=3y^2+2y+\cos(x+y)$$

所以 $k'(y)=3y^2+2y \quad \Rightarrow \quad k(y)=\int(3y^2+2y)dy+\tilde{c}=y^3+y^2+\tilde{c}$

因此　$u=\sin(x+y)+y^3+y^2+\tilde{c}=c^*$　$\Rightarrow$　$u=\sin(x+y)+y^3+y^2=c^*-\tilde{c}=c$

Example 2.　$(\cos y\sinh x+1)dx-\sin y\cosh xdy=0$　,　　$y(1)=2$

Sol.　　$M=\cos y\sinh x+1$,　　　　$N=-\sin y\cosh x$

$$\frac{\partial M}{\partial y}=-\sin y\sinh x,\qquad \frac{\partial N}{\partial x}=-\sin y\sinh x \quad\Rightarrow\quad \text{正合}$$

所以　$u=\int Ndy+l(x)=\int -\sin y\cosh xdy+l(x)=\cos y\cosh x+l(x)$

爲了要得到 $l(x)$ 將上式對 x 微分

$$\frac{\partial u}{\partial x}=\cos y\sinh x+l'(x)\equiv M=\cos y\sinh x+1$$

所以　$l'(x)=1$　$\Rightarrow$　$l(x)=\int 1\cdot dx+\tilde{c}=x+\tilde{c}$

因此　$u(x,y)=\cos y\cosh x+x=c$　將起始値代入　$\cos 2\cosh 1+1=c=0.358$
所以其解爲　$\cos y\cosh x+x=0.358$

Example 3.　　$-ydx+xdy=0$

Sol.　　$M=-y$,　$N=x$　　$\frac{\partial M}{\partial y}=-1\neq\frac{\partial N}{\partial x}=1$,　$\Rightarrow$　不是正合

$$u=\int Mdx+k(y)=-yx+k(y)\quad \text{因此}\quad \frac{\partial u}{\partial y}=-x+k'(y)\equiv N=x\quad \text{（不合）}$$

◎ **積分因子**(integrating factor)：假如一 ODE　$P(x,y)dx+Q(x,y)dy=0$　不是正合，但是　$F(x,y)P(x,y)dx+F(x,y)Q(x,y)dy=0$　是正合，則　$F(x,y)$　爲此 ODE 的積分因子

Example 4.

$$d\left(\frac{x}{y}\right)=-\frac{-ydx+xdy}{y^2}\qquad \text{積分因子}\quad \frac{1}{y^2}$$

$$d\left(\frac{y}{x}\right)=\frac{-ydx+xdy}{x^2}\qquad \text{積分因子}\quad \frac{1}{x^2}$$

$$d\ln\left(\frac{y}{x}\right)=\frac{-ydx+xdy}{xy}\qquad \text{積分因子}\quad \frac{1}{xy}$$

$$d\tan^{-1}\left(\frac{y}{x}\right)=\frac{-ydx+xdy}{x^2+y^2}\qquad \text{積分因子}\quad \frac{1}{x^2+y^2}$$

◎ **如何找到積分因子**　　　假如　$FPdx+FQdy=0$　是正合，則

$$\frac{\partial}{\partial y}(FP)=\frac{\partial}{\partial x}(FQ) \qquad \Rightarrow \qquad F_yP+FP_y=F_xQ+FQ_x$$

(1) 假如讓 $F=F(x)$，則 $F_y=0,\ F_x=F'=\frac{dF}{dx}$，所以原式變爲

$$FP_y=F'Q+FQ_x \qquad \Rightarrow \quad \frac{1}{F}\frac{dF}{dx}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=R$$

假如 $\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$ 是 x 的函數，則 $F(x)=\exp\int R(x)dx=e^{\int R(x)dx}$

(2) 假如讓 $F^*=F^*(y)$，則 $F_x^*=0,\ F_y^*=F^{*\prime}=\frac{dF^*}{dy}$，所以原式變爲

$$F^{*\prime}P+F^*P_y=F^*Q_x \quad \Rightarrow \quad \frac{1}{F^*}\frac{dF^*}{dy}=\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=R^*$$

假如 $\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$ 是 y 的函數，則 $F^*(y)=\exp\int R^*(y)dy=e^{\int R^*(y)dy}$

Example 5. $\left(e^{x+y}+ye^y\right)dx+\left(xe^y-1\right)dy=0,\quad y(0)=-1$

Sol. $P=e^{x+y}+ye^y,\qquad Q=xe^y-1$

$$\frac{\partial P}{\partial y}=e^{x+y}+e^y+ye^y,\quad \frac{\partial Q}{\partial x}=e^y,\quad \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x} \qquad \Rightarrow \ \text{不是正合}$$

$$R=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{xe^y-1}\left(e^{x+y}+e^y+ye^y-e^y\right)=R(x,y)$$

$$R^*=\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\frac{1}{e^{x+y}+ye^y}\left(e^y-e^{x+y}-e^y-ye^y\right)=-1=R^*(y)$$

所以積分因子 $F^*(y)=\exp\int R^*(y)dy=e^{\int R^*(y)dy}=e^{\int -1dy}=e^{-y}$

因此正合方程式爲 $F^*Pdx+F^*Qdy=\left(e^x+y\right)dx+\left(x-e^{-y}\right)dy=0$

$$\frac{\partial M}{\partial y}=\frac{\partial\left(e^x+y\right)}{\partial y}=1\equiv\frac{\partial N}{\partial x}=\frac{\partial\left(x-e^{-y}\right)}{\partial x}=1 \quad \Rightarrow \ \text{正合}$$

所以 $u=\int Mdx+k(y)=\int\left(e^x+y\right)dx+k(y)=e^y+xy+k(y)$

爲了要得到 $k(y)$ 將上式對 y 微分

$$\frac{\partial u}{\partial y}=x+k'(y)\equiv N=x-e^{-y} \ \text{所以}\ k'(y)=-e^{-y} \ \Rightarrow \ k(y)=e^{-y}+\tilde{c}$$

因此 $u(x,y)=e^x+xy+e^{-y}=c$ 將起始値 $y(0)=-1$ 代入

得 $e^0+0\cdot(-1)+e^{-(-1)}=c=1+e=3.72$

所以其解爲 $e^x+xy+e^{-y}=3.72$