Chapter 9

The Finite Volume Method for Diffusion Problems

Outline

- 9-1 Introduction to FVM
- 9-2 FVM for 1-D steady State Diffusion
- 9-3 FVM for 2-D steady State Diffusion
- 9-4 FVM for 3-D steady State Diffusion
- 9-5 Unsteady Term

9-1 Introduction to FVM (1)

- The general differential equation, for the conservation of a physical property, ϕ

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla.(\rho V\phi) = \nabla.(\Gamma\nabla\phi) + S$$

- The 4 terms are: Unsteady term, Convection term, Diffusion term and Source term
- In general, $\phi = \phi(x, y, z, t)$
- Γ is the diffusion coefficient corresponding to the particular property ϕ, S is the corresponding source term
- As ϕ takes different values we get conservation equations for different quantities

  eg: ϕ=1: Mass conservation

  ϕ=u: x-momentum conservation

  ϕ=h: Energy conservation

9-1 Introduction to FVM (2)

- Key concept: Integration of differential equation over Control Volume

$$\int_{CV}\left(\int_{t}^{t+\Delta t}\frac{\partial}{\partial t}(\rho\phi)dt\right)dV+\int_{t}^{t+\Delta t}\left(\int_{A}n.(\rho V\phi)\,dA\right)dt=\int_{t}^{t+\Delta t}\left(\int_{A}n.(\Gamma\nabla\phi)dA\right)dt+\int_{t}^{t+\Delta t}\int_{CV}S_{\phi}dV\,dt$$

- For simplification, we first do finite volume formulation for 1-D steady state equation(with no source term)

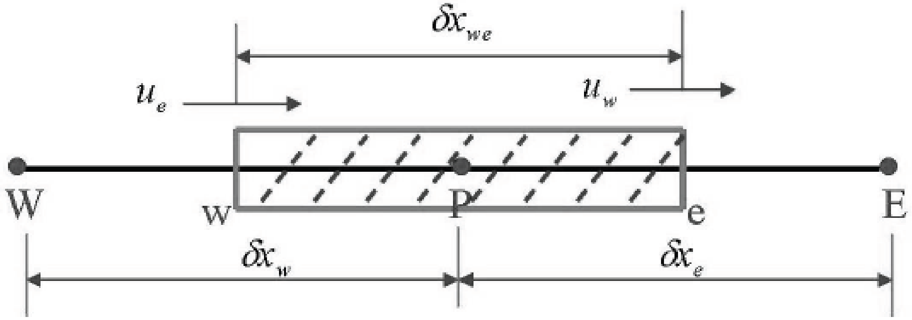
$$\frac{d}{dx}(\rho u\phi)=\frac{d}{dx}(\Gamma\frac{d\phi}{dx})$$

- The flow field should also satisfy continuity equation

$$\frac{d(\rho u)}{dx}=0\ or\ \rho u=constant$$

9-1 Introduction to FVM (3)

- Control Volume(CV) to be used:



- Integration of transport equation for the shown CV gives

$$(\rho u A \phi)_e - (\rho u A \phi)_w = (\Gamma A \frac{\partial \phi}{\partial x})_e - (\Gamma A \frac{\partial \phi}{\partial x})_w$$

- Derivatives for diffusion term are calculated assuming piecewise linear profile of

$$(\Gamma A \frac{\partial \phi}{\partial x})_e = \Gamma_e A_e (\frac{\phi_E - \phi_P}{\delta x_e}) \ , \ (\Gamma A \frac{\partial \phi}{\partial x})_w = \Gamma_w A_w (\frac{\phi_P - \phi_W}{\delta x_w})$$

9-1 Introduction to FVM (4)

- Assuming $A_e = A_w$, the integral of transport equation becomes,

$$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

where, $$F = \rho u \;\; and \;\; D = \frac{\Gamma}{\delta x}$$

$$F_w = (\rho u)_w \;\; , \quad F_e = (\rho u)_e$$

$$D_w = \frac{\Gamma_w}{\delta x_w}, \qquad D_e = \frac{\Gamma_e}{\delta x_e}$$

- Also, from continuity equation, we have

$$F_e = F_w$$

- There are various methods to calculate the Convection term and will be discussed after the four basic rules

9-1 Introduction to FVM (5)

- For solutions to be: 1. Physically realistic

  2. Satisfy overall balance (conservative)

  There are some basic rules that need to be satisfied by the discretization equations

Standard form of discretization equations(1-D): $a_P\phi_P = a_W\phi_W + a_E\phi_E + b$

Rule 1: Flux consistency at CV faces

When a face is common to two adjacent control volumes, flux across it must be represented by the same expression in discretization equations for both the control volumes

Rule 2: Positive coefficients

All coefficients must always be of same sign because an increase in ϕ_W or ϕ_E must lead to increase in ϕ_P

9-1 Introduction to FVM (6)

Rule 3: Negative slope linearization of source term

If source term is dependent on $\boxed{A}$, it is linearized as:

$$\bar{S} = S_C + S_P\phi_P$$

This S_P will then appear in a_P along with other terms. To ensure a_P remains positive, S_P must be negative or zero

Rule 4: Sum of neighbour coefficients

If governing differential equation contains only derivatives of $\boxed{A}$, both $\boxed{A}$ and $\boxed{A}$+c will satisfy the equation. In this case,

$$a_P = \sum a_{nb}$$

9-2 FVM for 1-D steady State Diffusion (1)

- As first approach the simplest transport process: pure diffusion in the steady state.
- Governing equation

$$\nabla \cdot (\Gamma \nabla \phi) + S_\phi = 0$$

- Integration in each CV

$$\int_{CV} \nabla \cdot (\Gamma \nabla \phi)\, dV + \int_{CV} S_\phi dV =$$

$$\int_{\partial CV} \mathbf{n} \cdot (\Gamma \nabla \phi)\, dA + \int_{CV} S_\phi dV = 0$$

FVM for 1-D steady state diffusion

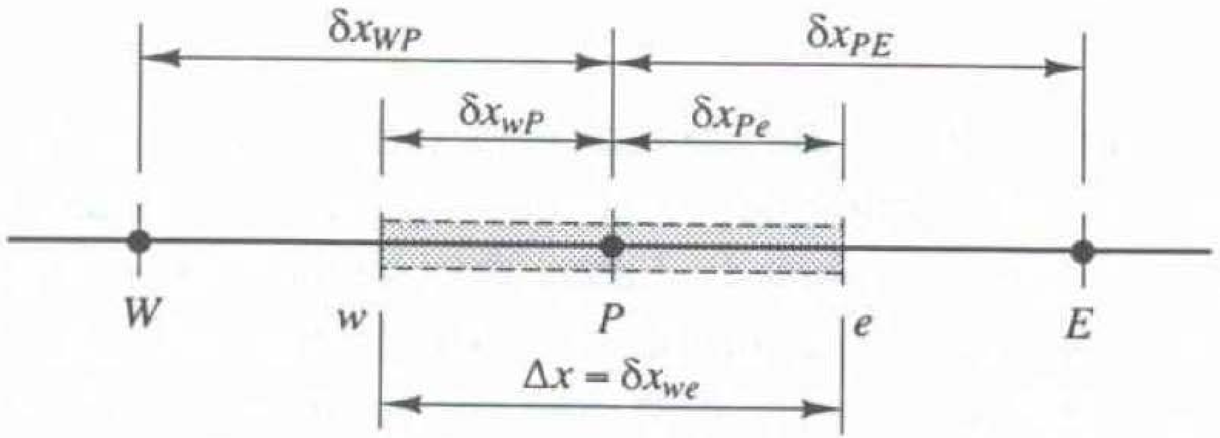
Differential equation to be considered:

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right) + S = 0$$

Boundary values must be prescribed for ϕ

9-2 FVM for 1-D steady State Diffusion (3)

Step 1: Grid generation



- First: divide the domain into discrete CV in such a way that physical boundary coincide with CV boundaries.
- Establish the standar notation (see figure)

9-2 FVM for 1-D steady State Diffusion (4)

Step 2: discretisation

$$\int_{CV} \nabla \cdot (\Gamma \nabla \phi)\, dV + \int_{CV} S_\phi dV =$$
$$\left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w + \overline{S}\Delta V = 0 \qquad (9)$$

- $\overline{S}$ average value of source S_ϕ over the CV.
- A cross-sectional area of the CV face
- Eq. (9) has a clear meaning: it states that the diffusive flux of ϕ leaving the east face minus the flux entering the west face is equal to the generation of ϕ inside the CV.

9-2 FVM for 1-D steady State Diffusion (5)

Step 2: discretisation

- Estimation of Γ and gradient of ϕ at faces e and w.
- ϕ and Γ are defined in the nodes. Gradients in the faces are calculated interpolating values between nodal points.
- For diffusive term, linear interpolation is the simplest manner (central differences). In a **uniform grid**

$$\Gamma_w = \frac{\Gamma_W + \Gamma_P}{2}; \qquad \Gamma_e = \frac{\Gamma_P + \Gamma_E}{2}$$

$$\left(\Gamma A \frac{d\phi}{dx}\right)_e = \Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right)$$

$$\left(\Gamma A \frac{d\phi}{dx}\right)_w = \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right)$$

9-2 FVM for 1-D steady State Diffusion (6)

Step 2: discretisation

- S may be a function of ϕ, so FVM approximates it by the linear expression

$$\overline{S}\Delta V = S_u + S_p\phi_p$$

- The discretised equation is then

$$\left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) - \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) + S_u + S_p\phi_p = 0$$

$$\Downarrow$$

$$\left(\frac{\Gamma_e A_e}{\delta x_{PE}} + \frac{\Gamma_w A_w}{\delta x_{WP}} - S_p\right)\phi_p =$$

$$\left(\frac{\Gamma_e A_e}{\delta x_{PE}}\right)\phi_E + \left(\frac{\Gamma_w A_w}{\delta x_{WP}}\right)\phi_W + S_u$$

9-2 FVM for 1-D steady State Diffusion (7)

Step 2: discretisation

- In a symbolic way it can be written:

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + S_u \qquad (11)$$

a_W	a_E	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$a_W + a_E - S_p$

Tri-diagonal matrix

- This type of discretisation is central to all further developments

9-2 FVM for 1-D steady State Diffusion (8)

Step 3: Solution of equations

- Eq. (11) must be set up for each nodal point.

- It must be modified in CV adjacent to the domain boundaries to incorporate the boundary conditions.

- Resulting system of linear algebraic equations has to be solved to obtain the distribution of ϕ at nodes.

- There exist numerical algorithms specially designed for CFD.

9-2 FVM for 1-D steady State Diffusion (9)

Thomas' Algorithm

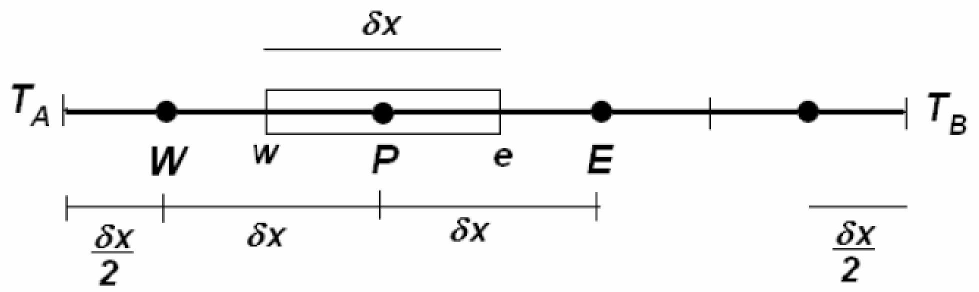
Gaussian elimination can be efficiently applied to a non-singular tridiagonal system of the form

$$\begin{pmatrix} \alpha_1 & \gamma_1 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & \gamma_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \ddots & \beta_{n-1} & \alpha_{n-1} & \gamma_{n-1} \\ 0 & \cdots & 0 & \beta_n & \alpha_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

using the following algorithm:

$$\begin{aligned}
&\delta_1 = \alpha_1 \\
&c_1 = b_1 \\
&\text{for } k = 2, 3, \ldots, n \qquad \text{(upper triangular form)} \\
&\qquad m_k = \beta_k / \delta_{k-1} \\
&\qquad \delta_k = \alpha_k - m_k \gamma_{k-1} \\
&\qquad c_k = b_k - m_k c_{k-1} \\
&v_n = c_n / \delta_n \qquad\qquad \text{(backsubstitution)} \\
&\text{for } k = n-1, n-2, \ldots, 1 \\
&\qquad v_k = (c_k - \gamma_k v_{k+1}) / \delta_k
\end{aligned}$$

Boundary conditions



T_A and T_B fixed temperatures (Dirichlet boundary conditions)

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+q=0 \qquad \left[k_e A\left(\frac{T_E-T_P}{\delta x}\right)-k_w A\left(\frac{T_P-T_W}{\delta x}\right)\right]+qA\delta x=0$$

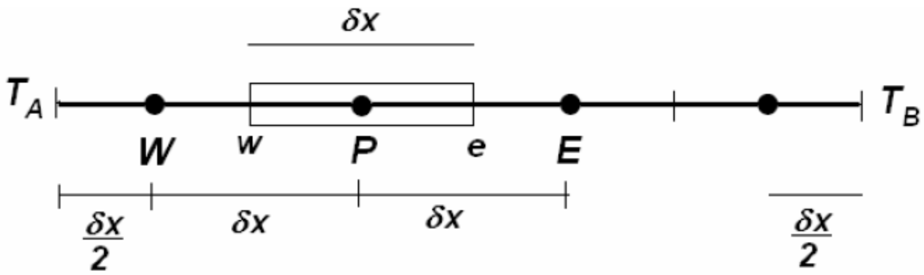
After discretization $\quad a_P T_P = a_W T_W + a_E T_E + S_u$

a_W	a_E	a_P	S_p	S_u
$kA/\delta x$	$kA/\delta x$	$a_W + a_E - S_p$	0	$qA\delta x$

Valid for the inner CV's

9-2 FVM for 1-D steady State Diffusion (11)

Boundary conditions



Left CV:

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_A A\left(\frac{T_P - T_A}{\delta x/2}\right)\right] + qA\delta x = 0$$

a_W	a_E	a_P	S_p	S_u
0	$kA/\delta x$	$a_W + a_E - S_p$	$-2kA/\delta x$	$qA\delta x+2kAT_A/\delta x$

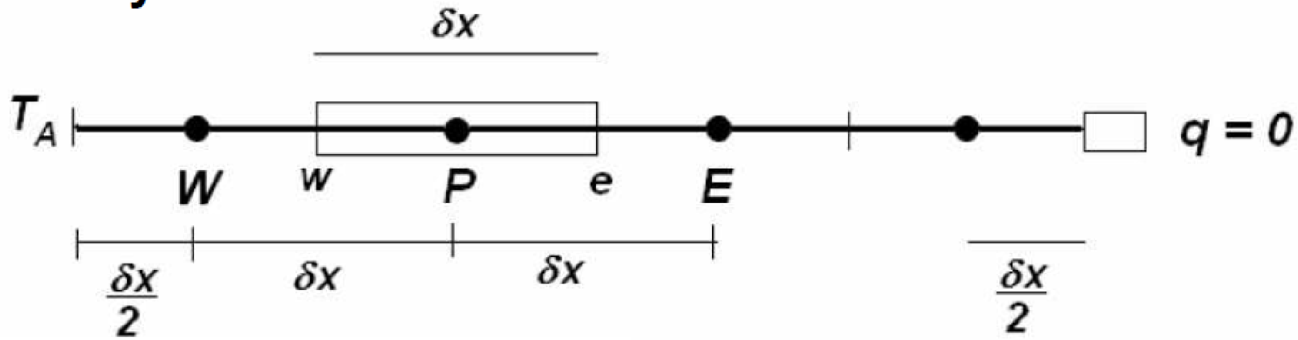
Right CV:

$$\left[k_B A\left(\frac{T_B - T_P}{\delta x/2}\right) - k_w A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0$$

a_W	a_E	a_P	S_p	S_u
$kA/\delta x$	0	$a_W + a_E - S_p$	$-2kA/\delta x$	$qA\delta x+2kAT_B/\delta x$

9-2 FVM for 1-D steady State Diffusion (12)

Boundary conditions



T_A fixed and insulated bc at the right side (zero heat flux)

Inner CV's (k, A, h constants)

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0$$

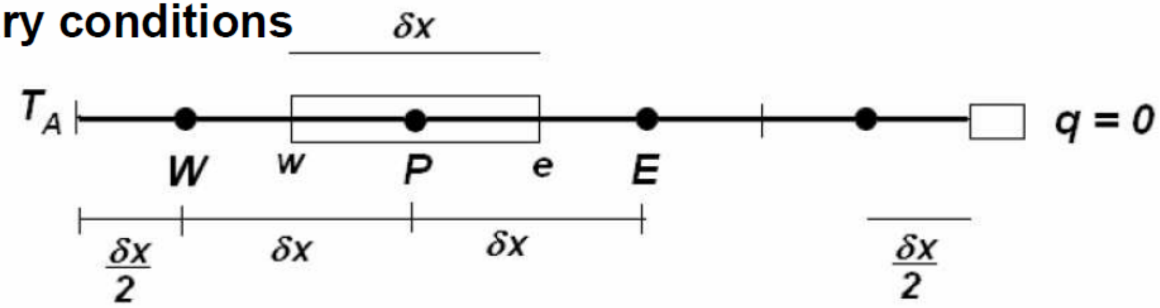
Define $n^2 = hP/(kA)$

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_W}{\delta x}\right)\right] - \left[n^2(T_P - T_\infty)\delta x\right] = 0$$

In the general form: $a_P T_P = a_W T_W + a_E T_E + S_u$

a_W	a_E	a_P	S_p	S_u
$1/\delta x$	$1/\delta x$	$a_W + a_E - S_p$	$-n^2\delta x$	$n^2 T_\infty \delta x$

Boundary conditions



Left CV:

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_A}{\delta x/2}\right)\right] - \left[n^2(T_P - T_\infty)\delta x\right] = 0$$

a_W	a_E	a_P	S_p	S_u
0	$1/\delta x$	$a_W + a_E - S_p$	$-n^2\delta x - 2/\delta x$	$n^2 T_\infty \delta x + 2T_A/\delta x$

Right CV:

$$\left[0 - \left(\frac{T_P - T_W}{\delta x}\right)\right] - \left[n^2(T_P - T_\infty)\delta x\right] = 0$$

a_W	a_E	a_P	S_p	S_u
$1/\delta x$	0	$a_W + a_E - S_p$	$-n^2\delta x$	$n^2 T_\infty \delta x$

9-2 FVM for 1-D steady State Diffusion (14)

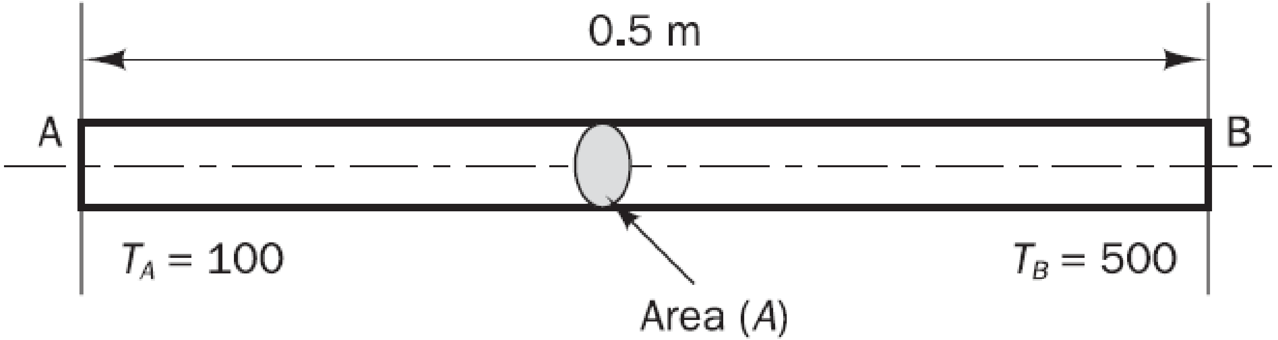
Example 4.1

Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100°C and 500°C respectively. The one-dimensional problem sketched in Figure 4.3 is governed by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{4.13}$$

Calculate the steady state temperature distribution in the rod. Thermal conductivity k equals 1000 W/m.K, cross-sectional area A is 10×10^{-3} m^2.

Figure 4.3



Solution Let us divide the length of the rod into five equal control volumes as shown in Figure 4.4. This gives $\delta x = 0.1$ m.

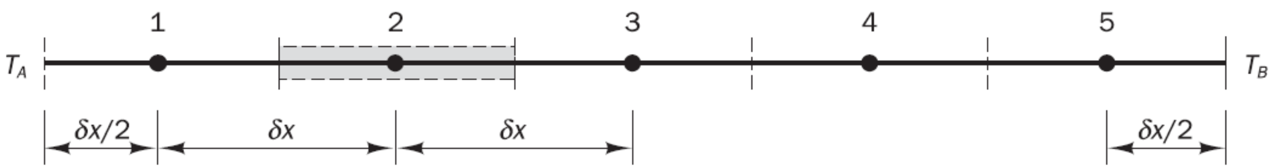


Figure 4.4 The grid used

The grid consists of five nodes. For each one of nodes 2, 3 and 4 temperature values to the east and west are available as nodal values. Consequently, discretised equations of the form (4.10) can be readily written for control volumes surrounding these nodes:

$$\left(\frac{k_e}{\delta x_{PE}}A_e + \frac{k_w}{\delta x_{WP}}A_w\right)T_P = \left(\frac{k_w}{\delta x_{WP}}A_w\right)T_W + \left(\frac{k_e}{\delta x_{PE}}A_e\right)T_E \qquad (4.14)$$

9-2 FVM for 1-D steady State Diffusion (16)

The thermal conductivity ($k_e = k_w = k$), node spacing (δx) and cross-sectional area ($A_e = A_w = A$) are constants. Therefore the **discretised equation for nodal points 2, 3 and 4** is

$$\boxed{a_P T_P = a_W T_W + a_E T_E} \tag{4.15}$$

with

a_W	a_E	a_P
$\frac{k}{\delta x}A$	$\frac{k}{\delta x}A$	$a_W + a_E$

S_u and S_p are zero in this case since there is no source term in the governing equation (4.13).

9-2 FVM for 1-D steady State Diffusion (17)

Nodes 1 and 5 are boundary nodes, and therefore require special attention. Integration of equation (4.13) over the control volume surrounding point 1 gives

$$kA\left(\frac{T_E - T_P}{\delta x}\right) - kA\left(\frac{T_P - T_A}{\delta x/2}\right) = 0 \qquad (4.16)$$

This expression shows that the flux through control volume boundary A has been approximated by assuming a linear relationship between temperatures at boundary point A and node P. We can rearrange (4.16) as follows:

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = 0\,.\,T_W + \left(\frac{k}{\delta x}A\right)T_E + \left(\frac{2k}{\delta x}A\right)T_A \qquad (4.17)$$

Comparing equation (4.17) with equation (4.10), it can be easily identified that the fixed temperature boundary condition enters the calculation as a source term $(S_u + S_P T_P)$ with $S_u = (2kA/\delta x)T_A$ and $S_p = -2kA/\delta x$, and that the link to the (west) boundary side has been suppressed by setting coefficient a_W to zero.

9-2 FVM for 1-D steady State Diffusion (18)

Equation (4.17) may be cast in the same form as (4.11) to yield the **discretised equation for boundary node 1**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.18}$$

with

a_W	a_E	a_P	S_P	S_u
0	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_p$	$-\dfrac{2kA}{\delta x}$	$\dfrac{2kA}{\delta x}T_A$

The control volume surrounding node 5 can be treated in a similar manner. Its discretised equation is given by

$$kA\left(\frac{T_B - T_P}{\delta x/2}\right) - kA\left(\frac{T_P - T_W}{\delta x}\right) = 0 \tag{4.19}$$

As before we assume a linear temperature distribution between node P and boundary point B to approximate the heat flux through the control volume boundary. Equation (4.19) can be rearranged as

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = \left(\frac{k}{\delta x}A\right)T_W + 0\,.\,T_E + \left(\frac{2k}{\delta x}A\right)T_B \tag{4.20}$$

9-2 FVM for 1-D steady State Diffusion (19)

The **discretised equation for boundary node 5** is

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.21}$$

where

a_W	a_E	a_P	S_P	S_u
$\dfrac{kA}{\delta x}$	0	$a_W + a_E - S_p$	$-\dfrac{2kA}{\delta x}$	$\dfrac{2kA}{\delta x} T_B$

The discretisation process has yielded one equation for each of the nodal points 1 to 5. Substitution of numerical values gives $kA/\delta x = 100$, and the coefficients of each discretised equation can easily be worked out. Their values are given in Table 4.1.

9-2 FVM for 1-D steady State Diffusion (20)

Table 4.1

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	100	$200T_A$	–200	300
2	100	100	0	0	200
3	100	100	0	0	200
4	100	100	0	0	200
5	100	0	$200T_B$	–200	300

The resulting set of algebraic equations for this example is

$$300T_1 = 100T_2 + 200T_A$$
$$200T_2 = 100T_1 + 100T_3$$
$$200T_3 = 100T_2 + 100T_4$$
$$200T_4 = 100T_3 + 100T_5$$
$$300T_5 = 100T_4 + 200T_B$$

9-2 FVM for 1-D steady State Diffusion (21)

This set of equations can be rearranged as

$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{bmatrix} \tag{4.23}$$

The above set of equations yields the steady state temperature distribution of the given situation. For simple problems involving a small number of nodes the resulting matrix equation can easily be solved with a software package such as MATLAB (1992). For $T_A = 100$ and $T_B = 500$ the solution of (4.23) can obtained by using, for example, Gaussian elimination:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix} \tag{4.24}$$

9-2 FVM for 1-D steady State Diffusion (22)

The exact solution is a linear distribution between the specified boundary temperatures: $T = 800x + 100$. Figure 4.5 shows that the exact solution and the numerical results coincide.

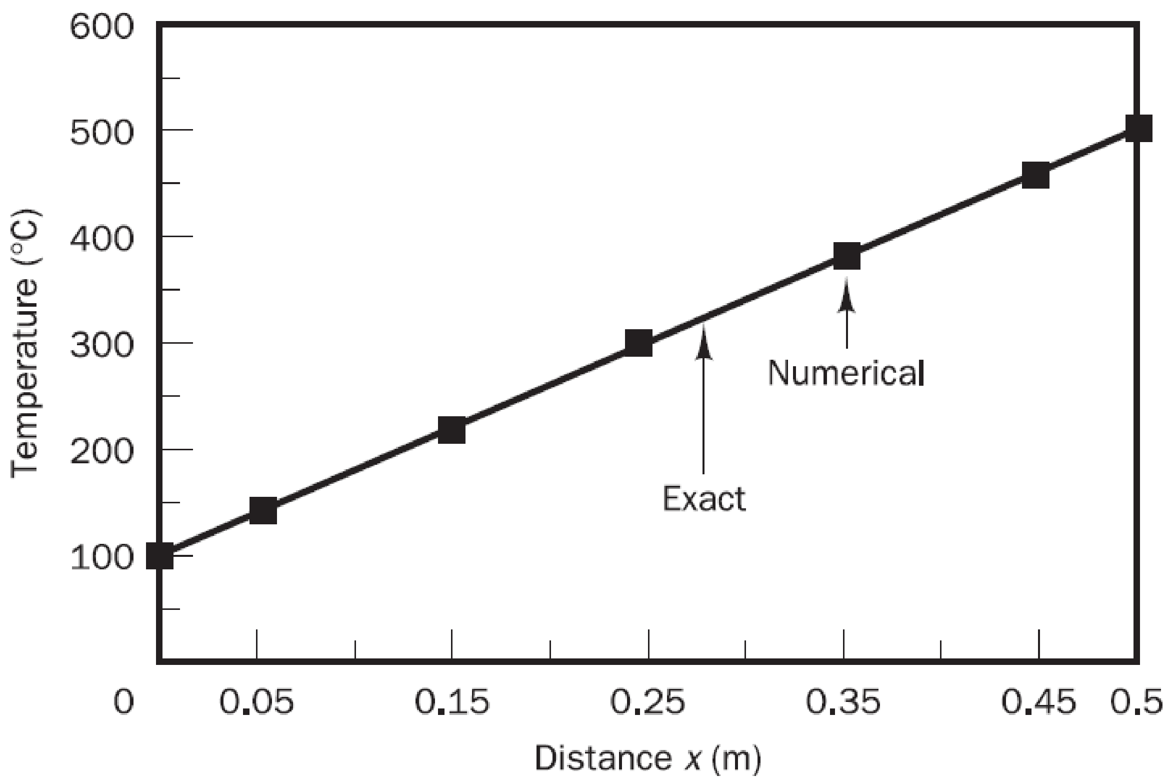
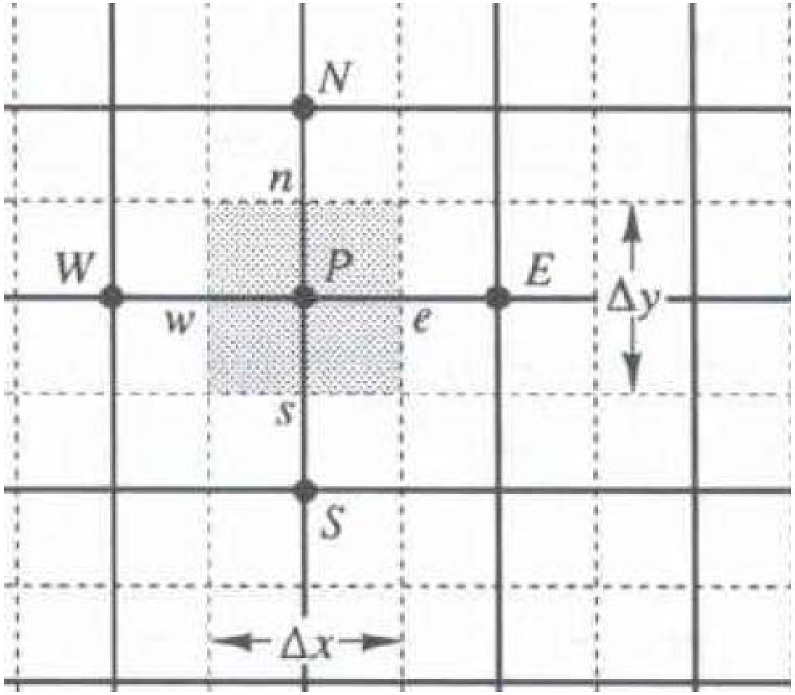


Figure 4.5

FVM for 2-D diffusion problems

- The methodology can be easily extended to 2-D

Grid



FVM for 2-D diffusion problems

- The methodology can be easily extended to 2-D. Eq. for 2-D steady state diffusion

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)+S=0$$

- Formal integration over the CV

$$\int_{CV}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)dxdy+\int_{CV}\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)dxdy+\int_{CV}S_\phi dxdy=0$$

$$\int_{CV} \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) dxdy + \int_{CV} \frac{\partial}{\partial y}\left(\Gamma \frac{\partial \phi}{\partial y}\right) dxdy + \int_{CV} S_\phi dxdy = 0$$

- As $A_e = A_w = \Delta y$ and $A_n = A_s = \Delta x$

$$\left[\Gamma_e A_e \left(\frac{\partial \phi}{\partial x}\right)_e - \Gamma_w A_w \left(\frac{\partial \phi}{\partial x}\right)_w\right] +$$

$$\left[\Gamma_n A_n \left(\frac{\partial \phi}{\partial y}\right)_n - \Gamma_s A_s \left(\frac{\partial \phi}{\partial y}\right)_s\right] + \overline{S}\Delta V = 0$$

- Discretisation of the derivatives as in 1-D, e.g.

$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{\delta x_{PE}}; \quad \left(\frac{\partial \phi}{\partial y}\right)_s = \frac{\phi_P - \phi_S}{\delta y_{SP}}$$

9-3 FVM for 2-D steady State Diffusion (4)

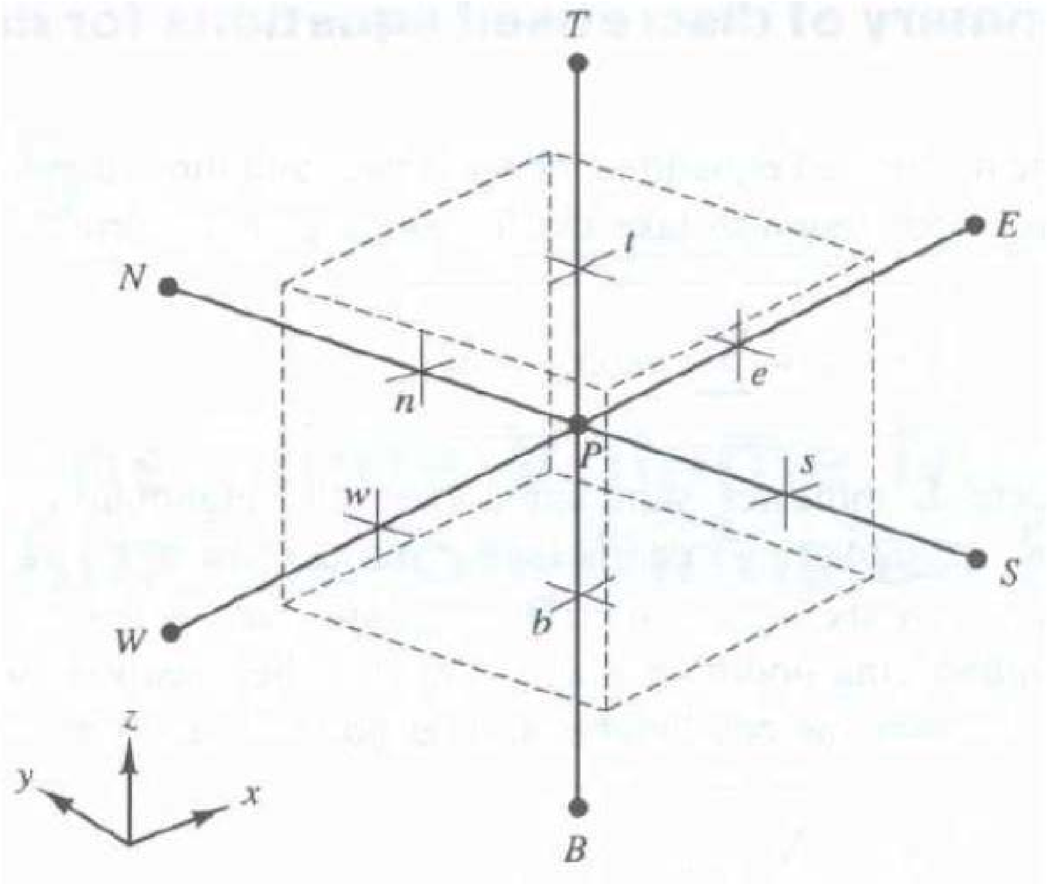
- Proceeding in the same way as in 1-D, it is obtained: Exercise!

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + S_u \qquad (12)$$

a_W	a_E	a_S	a_N	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$a_W + a_E + a_S + a_N - S_p$

- Eq. (12) is written for every node. Resulting set of algebraic equations.
- At boundaries the discretised eqs. are modified to introduce the boundary conditions.
- The final set of linear equations is solved to get the 2D distribution of ϕ.

FVM for 3-D diffusion problems



9-4 FVM for 3-D steady State Diffusion (2)

FVM for 3-D diffusion problems

- Steady state diffusion in 3-D

$$\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right)+S=0$$

- Algebraic equation for each CV

Exercise!

$$a_P\phi_P=a_E\phi_E+a_W\phi_W+a_N\phi_N+a_S\phi_S+a_B\phi_B+a_T\phi_T+S_u$$

a_W	a_E	a_S	a_N	a_B	a_T	a_P
$\frac{\Gamma_w A_w}{\delta x_{WP}}$	$\frac{\Gamma_e A_e}{\delta x_{PE}}$	$\frac{\Gamma_s A_s}{\delta y_{SP}}$	$\frac{\Gamma_n A_n}{\delta y_{PN}}$	$\frac{\Gamma_b A_b}{\delta z_{BP}}$	$\frac{\Gamma_t A_t}{\delta z_{PT}}$	$a_W+a_E+a_S+a_N$ $+a_B+a_T-S_p$

9-5 Unsteady Term (1)

- For handling the unsteady term we will look at 1-D unsteady equation without convection(without source term), later we can extend the concept to convection-diffusion equations of all kinds

$$\frac{\partial(\rho\phi)}{\partial t} = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)$$

- Integration over the 1-D CV gives

$$\int_{w}^{e}\int_{t}^{t+\Delta t}\frac{\partial(\rho\phi)}{\partial t}dtdx = \int_{t}^{t+\Delta t}\int_{w}^{e}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)dxdt$$

9-5 Unsteady Term (2)

- Density remains constant(from continuity equation)

$$\rho_P(\phi_P - \phi_P^o)\Delta x = \int_t^{t+\Delta t} \left(\Gamma_e \frac{\phi_E - \phi_P}{\delta x_e} - \Gamma_w \frac{\phi_P - \phi_W}{\delta x_w} \right) dt$$

- Now we need an assumption for ϕ with t, We assume

$$\int_t^{t+\Delta t} \phi_P dt = (f\phi_P + (1-f)\phi_P^o)\Delta t, \qquad 0 < f < 1$$

- We use similar formulas for ϕ_E and ϕ_W

9-5 Unsteady Term (3)

- Final Discretization Equation:

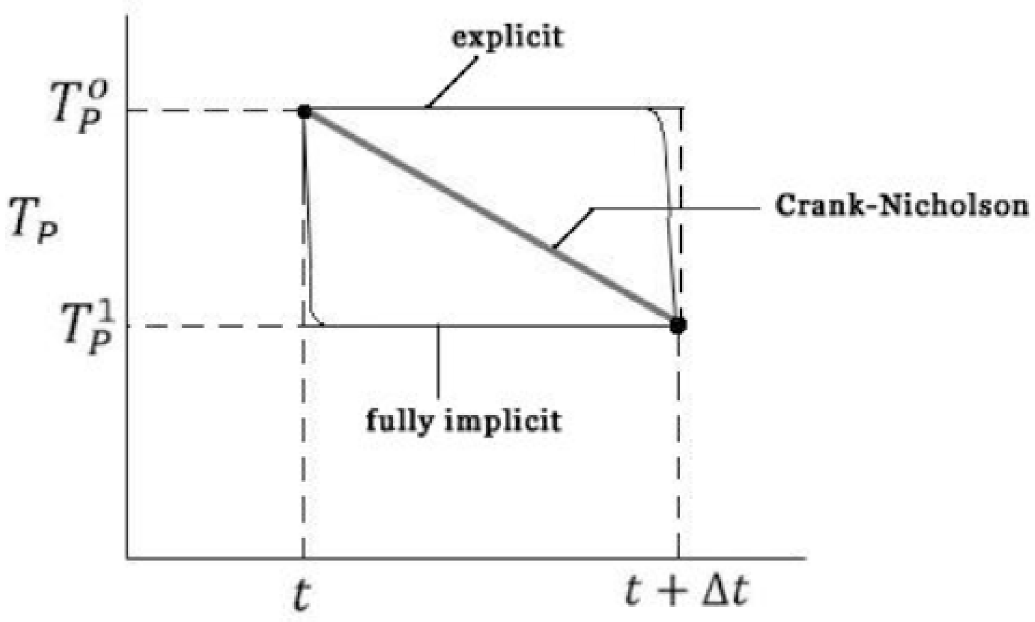
$$a_P\phi_P = a_E(f\phi_E + (1-f)\phi_E^0) + a_W(f\phi_W + (1-f)\phi_W^0) + (a_P^o - (1-f)a_E - (1-f)a_W)\phi_P^o$$

where,

a_E	a_W	a_P^o	a_P
$\dfrac{\Gamma_e}{\delta x_e}$	$\dfrac{\Gamma_w}{\delta x_w}$	$\dfrac{\rho_P \Delta x}{\Delta t}$	$f a_E + f a_W + a_P^o$

9-5 Unsteady Term (4)

- If f=0: Scheme is explicit
- If f=0.5: Crank Nicholson Scheme
- If f=1: Implicit Scheme
- Variation of Temperature with time for the three schemes is :



9-5 Unsteady Term (5)

Analysis:

- Explicit Scheme: $a_P\phi_P = a_E\phi_E^0 + a_W\phi_W^0 + (a_P^o - a_E - a_W)\phi_P^o$
- The coefficient of ϕ_P^o becomes negative if a_P^o exceeds $a_E + a_W$
- For uniform conductivity and equal grid spacing, scheme is stable if

$$\Delta t < \frac{\rho_P(\Delta x)^2}{2\Gamma}$$

- Crank Nicholson Scheme: Coefficient of ϕ_P^o is

$$a_P^o - \frac{(a_E + a_W)}{2} = \frac{\rho_P\Delta x}{\Delta t} - \frac{\Gamma}{\delta x}$$

9-5 Unsteady Term (6)

- Even in Crank- Nicholson Scheme, if the time step is not sufficiently small, the coefficient of ϕ_P^o will become negative
- Crank Nicholson Scheme is also conditionally stable

- Implicit Scheme: Only in this case, the coefficient of ϕ_P^o is always positive. Thus, fully implicit scheme satisfies requirements of simlicity and physically realistic behavior.

- However, at small time steps, Crank Nicholson scheme is more accurate than fully implicit scheme
- Reason: Temperature time curve is nearly linear for small time intervals which is exactly what we assumed in Crank Nicholson scheme