3-7 FLUIDS IN RIGID-BODY MOTION

We are almost ready to begin studying fluids in motion (starting in Chapter 4), but first there is one category of fluid motion that can be studied using fluid statics ideas: rigid-body motion. As the name implies, this is motion in which the entire fluid moves as if it were a rigid body—individual fluid particles, although they may be in motion, are not deforming. This means that, as in the case of a static fluid, there are no shear stresses.

What kind of fluid flow has rigid-body motion? You recall from kinematics that rigid-body motion can be broken down into pure translation and pure rotation. For translation the simplest motion is constant velocity, which can always be converted to a fluid statics problem by a shift of coordinates. The other simple translational motion we can have is constant acceleration, which we will consider here (Example 3.9). In addition, we will consider motion consisting of pure constant rotation (Example 3.10). As in the case of the static fluid, we may apply Newton's second law of motion to determine the pressure field that results from a specified rigid-body motion.

In Section 3-1 we derived an expression for the forces due to pressure and gravity acting on a fluid particle of volume $d\forall$. We obtained

$$d\vec{F} = (-\nabla p + \rho\vec{g})d\forall$$

or

$$\frac{d\vec{F}}{d\forall} = -\nabla p + \rho\vec{g} \tag{3.2}$$

Newton's second law was written

$$d\vec{F} = \vec{a}\,dm = \vec{a}\rho\,d\forall \quad \text{or} \quad \frac{d\vec{F}}{\forall} = \rho\vec{a}$$

Substituting from Eq. 3.2, we obtain

$$-\nabla p + \rho\vec{g} = \rho\vec{a} \tag{3.17}$$

If the acceleration $\vec{a}$ is constant, we can combine it with $\vec{g}$ and obtain an effective "acceleration of gravity," $\vec{g}_{eff} = \vec{g} - \vec{a}$, so that Eq. 3.17 has the same form as our basic equation for pressure distribution in a static fluid, Eq. 3.3:

$$-\nabla p + \rho\vec{g}_{eff} = 0 \qquad [\text{Compare to } -\nabla p + \rho\vec{g} = 0 \quad (3.3)]$$

This means that we can use the results of previous sections of this chapter as long as we use $\vec{g}_{eff}$ in place of $\vec{g}$. For example, for a liquid undergoing constant acceleration the pressure increases with depth in the direction of $\vec{g}_{eff}$, and the rate of increase of pressure will be given by ρg_{eff}, where g_{eff} is the magnitude of $\vec{g}_{eff}$. Lines of constant pressure will be perpendicular to the direction of $\vec{g}_{eff}$. The physical significance of each term in Eq. 3.17 is as follows:

$$\begin{matrix} -\nabla p & + & \rho\vec{g} & = & & \rho\vec{a} \\ \left\{\begin{matrix}\text{net pressure force}\\ \text{per unit volume}\\ \text{at a point}\end{matrix}\right\} & + & \left\{\begin{matrix}\text{body force per}\\ \text{unit volume}\\ \text{at a point}\end{matrix}\right\} & = & \left\{\begin{matrix}\text{mass per}\\ \text{unit}\\ \text{volume}\end{matrix}\right\} \times & \left\{\begin{matrix}\text{acceleration}\\ \text{of fluid}\\ \text{particle}\end{matrix}\right\} \end{matrix}$$

This vector equation consists of three component equations that must be satisfied individually. In rectangular coordinates the component equations are

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} + \rho g_x &= \rho a_x \qquad x \text{ direction} \\ -\frac{\partial p}{\partial y} + \rho g_y &= \rho a_y \qquad y \text{ direction} \\ -\frac{\partial p}{\partial z} + \rho g_z &= \rho a_z \qquad z \text{ direction} \end{aligned}\right\} \tag{3.18}$$

Component equations for other coordinate systems can be written using the appropriate expression for ∇p. In cylindrical coordinates the vector operator, ∇, is given by

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \tag{3.19}$$

where $\hat{e}_r$ and $\hat{e}_\theta$ are unit vectors in the r and θ directions, respectively. Thus

$$\nabla p = \hat{e}_r \frac{\partial p}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \hat{k}\frac{\partial p}{\partial z} \tag{3.20}$$

Example 3.9 Liquid in Rigid-Body Motion with Linear Acceleration

As a result of a promotion, you are transferred from your present location. You must transport a fish tank in the back of your minivan. The tank is 12 in. × 24 in. × 12 in. How much water can you leave in the tank and still be reasonably sure that it will not spill over during the trip?

GIVEN: Fish tank 12 in. × 24 in. × 12 in. partially filled with water to be transported in an automobile.

FIND: Allowable depth of water for reasonable assurance that it will not spill during the trip.

SOLUTION:

The first step in the solution is to formulate the problem by translating the general problem into a more specific one.

We recognize that there will be motion of the water surface as a result of the car's traveling over bumps in the road, going around corners, etc. However, we shall assume that the main effect on the water surface is due to linear accelerations (and decelerations) of the car; we shall neglect sloshing.

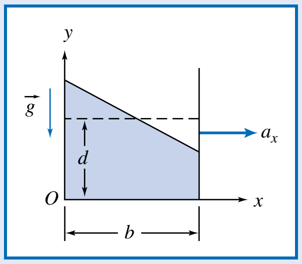
Thus we have reduced the problem to one of determining the effect of a linear acceleration on the free surface. We have not yet decided on the orientation of the tank relative to the direction of motion. Choosing the x coordinate in the direction of motion, should we align the tank with the long side parallel, or perpendicular, to the direction of motion?

If there will be no relative motion in the water, we must assume we are dealing with a constant acceleration, a_x. What is the shape of the free surface under these conditions?

Let us restate the problem to answer the original questions by idealizing the physical situation to obtain an approximate solution.

GIVEN: Tank partially filled with water (to depth d) subject to constant linear acceleration, a_x. Tank height is 12 in.; length parallel to direction of motion is b. Width perpendicular to direction of motion is c.

FIND: (a) Shape of free surface under constant a_x.
(b) Allowable water depth, d, to avoid spilling as a function of a_x and tank orientation.
(c) Optimum tank orientation and recommended water depth.



SOLUTION:

Governing equation: $-\nabla p + \rho\vec{g} = \rho\vec{a}$

$$-\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) + \rho(\hat{i}g_x + \hat{j}g_y + \hat{k}g_z) = \rho(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z)$$

Since p is not a function of z, $\partial p/\partial z = 0$. Also, $g_x = 0$, $g_y = -g$, $g_z = 0$, and $a_y = a_z = 0$.

$$\therefore -\hat{i}\frac{\partial p}{\partial x} - \hat{j}\frac{\partial p}{\partial y} - \hat{j}\rho g = \hat{i}\rho a_x$$

The component equations are:

$$\frac{\partial p}{\partial x} = -\rho a_x$$

$$\frac{\partial p}{\partial y} = -\rho g$$

{Recall that a partial derivative means that all other independent variables are held constant in the differentiation.}

The problem now is to find an expression for $p = p(x, y)$. This would enable us to find the equation of the free surface. But perhaps we do not have to do that.

Since the pressure is $p = p(x, y)$, the difference in pressure between two points (x, y) and $(x + dx, y + dy)$ is

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Since the free surface is a line of constant pressure, p = constant along the free surface, so $dp = 0$ and

$$0 = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = -\rho a_x\,dx - \rho g\,dy$$

Therefore,

$$\left.\frac{dy}{dx}\right)_{\text{free surface}} = -\frac{a_x}{g}$$

{The free surface is a plane.}

Note that we could have derived this result more directly by converting Eq. 3.17 into an equivalent "acceleration of gravity" problem,

$$-\nabla p + \rho \vec{g}_{eff} = 0$$

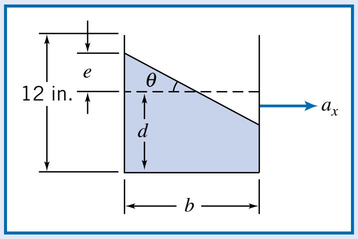
where $\vec{g}_{eff} = \vec{g} - \hat{i}a_x = -\hat{i}a_x - \hat{j}g$. Lines of constant pressure (including the free surface) will then be perpendicular to the direction of $\vec{g}_{eff}$, so that the slope of these lines will be $-1/(g/a_x) = -a_x/g$.

In the diagram,

d = original depth

e = height above original depth

b = tank length parallel to direction of motion



$$e = \frac{b}{2}\tan\theta = \frac{b}{2}\left(-\left.\frac{dy}{dx}\right)_{\text{free surface}}\right) = \frac{b}{2}\frac{a_x}{g}$$

{Only valid when the free surface intersects the front wall at or above the floor}

Since we want e to be smallest for a given a_x, the tank should be aligned so that b is as small as possible. We should align the tank with the long side perpendicular to the direction of motion. That is, we should choose $b = 12$ in. ← b

With $b = 12$ in.,

$$e = 6\frac{a_x}{g}\text{ in.}$$

The maximum allowable value of $e = 12 - d$ in. Thus

$$12 - d = 6\frac{a_x}{g} \quad \text{and} \quad d_{\max} = 12 - 6\frac{a_x}{g}$$

If the maximum a_x is assumed to be ⅔ g, then allowable d equals 8 in.

To allow a margin of safety, perhaps we should select $d = 6$ in. ← d

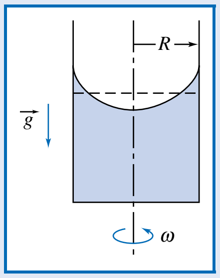
Recall that a steady acceleration was assumed in this problem. The car would have to be driven *very* carefully and smoothly.

This Example shows that:

- ✓ Not all engineering problems are clearly defined, nor do they have unique answers.
- ✓ For constant linear acceleration, we effectively have a hydrostatics problem, with "gravity" redefined as the vector result of the acceleration and the actual gravity.

Example 3.10 Liquid in Rigid-Body Motion with Constant Angular Speed

A cylindrical container, partially filled with liquid, is rotated at a constant angular speed, ω, about its axis as shown in the diagram. After a short time there is no relative motion; the liquid rotates with the cylinder as if the system were a rigid body. Determine the shape of the free surface.



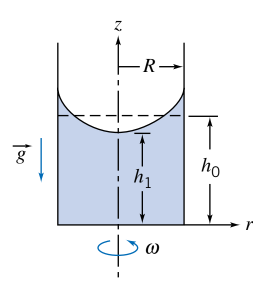
GIVEN: A cylinder of liquid in rigid-body rotation with angular speed ω about its axis.

FIND: Shape of the free surface.

SOLUTION:

Governing equation:

$$-\nabla p + \rho \vec{g} = \rho \vec{a}$$



It is convenient to use a cylindrical coordinate system, r, θ, z. Since $g_r = g_\theta = 0$ and $g_z = -g$, then

$$-\left(\hat{e}_r \frac{\partial p}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \hat{k}\frac{\partial p}{\partial z}\right) - \hat{k}\rho g = \rho(\hat{e}_r a_r + \hat{e}_\theta a_\theta + \hat{k} a_z)$$

Also, $a_\theta = a_z = 0$ and $a_r = -\omega^2 r$.

$$\therefore -\left(\hat{e}_r \frac{\partial p}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial p}{\partial \theta} + \hat{k}\frac{\partial p}{\partial z}\right) = -\hat{e}_r \rho \omega^2 r + \hat{k}\rho g$$

The component equations are:

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \qquad \frac{\partial p}{\partial \theta} = 0 \qquad \frac{\partial p}{\partial z} = -\rho g$$

From the component equations we see that the pressure is not a function of θ; it is a function of r and z only.

Since $p = p(r, z)$, the differential change, dp, in pressure between two points with coordinates (r, θ, z) and $(r + dr, \theta, z + dz)$ is given by

$$dp = \left.\frac{\partial p}{\partial r}\right)_z dr + \left.\frac{\partial p}{\partial z}\right)_r dz$$

Then

$$dp = \rho\omega^2 r dr - \rho g dz$$

To obtain the pressure difference between a reference point (r_1, z_1), where the pressure is p_1, and the arbitrary point (r, z), where the pressure is p, we must integrate

$$\int_{p_1}^{p} dp = \int_{r_1}^{r} \rho\omega^2 r\, dr - \int_{z_1}^{z} \rho g\, dz$$

$$p - p_1 = \frac{\rho\omega^2}{2}(r^2 - r_1^2) - \rho g(z - z_1)$$

Taking the reference point on the cylinder axis at the free surface gives

$$p_1 = p_{atm} \quad r_1 = 0 \quad z_1 = h_1$$

Then

$$p - p_{atm} = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

Since the free surface is a surface of constant pressure ($p = p_{atm}$), the equation of the free surface is given by

$$0 = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

or

$$z = h_1 + \frac{(\omega r)^2}{2g}$$

The equation of the free surface is a paraboloid of revolution with vertex on the axis at $z = h_1$.

We can solve for the height h_1 under conditions of rotation in terms of the original surface height, h_0, in the absence of rotation. To do this, we use the fact that the volume of liquid must remain constant. With no rotation

$$\forall = \pi R^2 h_0$$

With rotation

$$\forall = \int_0^R \int_0^z 2\pi r\, dz\, dr = \int_0^R 2\pi z r\, dr = \int_0^R 2\pi\left(h_1 + \frac{\omega^2 r^2}{2g}\right) r\, dr$$

$$\forall = 2\pi\left[h_1 \frac{r^2}{2} + \frac{\omega^2 r^4}{8g}\right]_0^R = \pi\left[h_1 R^2 + \frac{\omega^2 R^4}{4g}\right]$$

Then

$$\pi R^2 h_0 = \pi\left[h_1 R^2 + \frac{\omega^2 R^4}{4g}\right] \quad \text{and} \quad h_1 = h_0 - \frac{(\omega R)^2}{4g}$$

Finally,

$$z = h_0 - \frac{(\omega R)^2}{4g} + \frac{(\omega r)^2}{2g} = h_0 - \frac{(\omega R)^2}{2g}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right] \qquad z(r)$$

Note that the expression for z is valid only for $h_1 > 0$. Hence the maximum value of ω is given by $\omega_{max} = 2\sqrt{gh_0}/R$.

This Example shows:

- ✓ The effect of centripetal acceleration on the shape of constant pressure lines (isobars).
- ✓ Because the hydrostatic pressure variation and variation due to rotation each depend on fluid density, the final free surface shape is independent of fluid density.