OUTLINE

8-1 Stability considerations (1)

- At a starting point for stability analysis, consider the simple explicit approximation to the heat equation

$$\left(\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x^2}\right) \qquad \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) ----(1)$$

This may be solved for u^n_j to yield

$$u_j^{n+1} = u_j^n + \alpha \frac{\Delta t}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) ----(2)$$

8-1 Stability considerations (2)

Let D be the exact solution of this equation(2), N the numerical solution of equation(1) and A the analytical solution of the PDE:

$$\left(\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x^2} \right)$$

Then ,we may write

Discretization error=A-D

Round-off error=N-D

8-1 Stability considerations (3)

- The equation of stability of a numerical method examine the error growth while computations are being performed.

$$round-off \qquad error\left\langle \begin{array}{l} \quad Grow \Rightarrow instability \\ Nor \quad grow \Rightarrow stability \end{array} \right.$$

The equation of stability is usually answered by using a Fourier analysis. This method is also referred to as a von Neumann analysis.

8-2 Fourier or von Neumann analysis (1)

- Consider eq.(1) and let εbe the round-off error. The numerical solution actually computed may be written N=D+ε----(3)

  N must satisfy eq.(1). Substituting eq.(3) into eq.(1), yields

$$\frac{D_j^{n+1}+\varepsilon_j^{n+1}-D_j^n-\varepsilon_j^n}{\Delta t}=\alpha\left[\frac{D_{j+1}^{n+1}+\varepsilon_{j+1}^{n+1}-2D_j^n-2\varepsilon_j^n+D_{j-1}^n+\varepsilon_{j-1}^n}{(\Delta x)^2}\right]$$

8-2 Fourier or von Neumann analysis (2)

Since the exact solution must satisfy the difference eq.(i.e. Eq(1)), therefore

$$\frac{\varepsilon_j^{n+1}-\varepsilon_j^n}{\Delta t}=\alpha\left[\frac{\varepsilon_{j+1}^{n+1}-2\varepsilon_j^n+\varepsilon_{j-1}^n}{(\Delta x)^2}\right]$$

8-2 Fourier or von Neumann analysis (3)

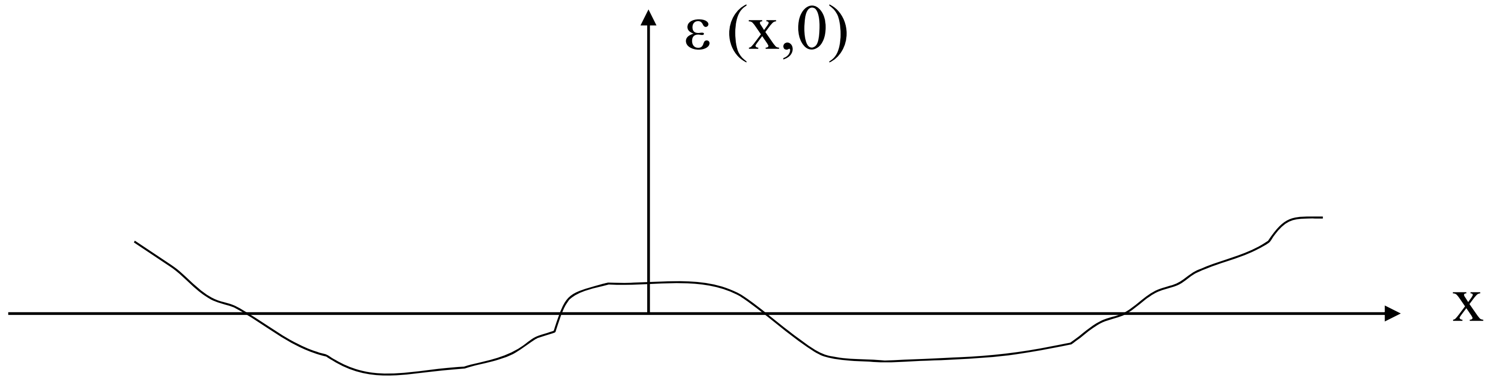
In this case, the exact solution D and the error ε must both satisfy the same difference equation. This means that the numerical error and exact numerical solution both posses the same growth property in time and either could be used to examine stability.

8-2 Fourier or von Neumann analysis (4)

Any perturbation of the input values at the n^{th} time level will either be prevented from growing without bound for a stable system or will grow large for an unstable system

8-2 Fourier or von Neumann analysis (5)

- Consider a distribution of error at any time in a mesh. We choose to view this distribution a time t=0 for convenience. This error distribution is shown



8-2 Fourier or von Neumann analysis (6)

- We assume the error ε (x,t) can be written as a series of the form

$$\varepsilon(x,t)=\sum_m b_m(t)e^{ik_m x} \quad ----(4)$$

$$e^{ikmx}=\cos k_m x+i\sin k_m x,(k_m : wave \quad number)$$

8-2 Fourier or von Neumann analysis (7)

Since the difference equation is linear, superposition may be used, and we may examine the behavior of a single term of the series given in eq.(4).

consider the term

$$\varepsilon\left(x,t\right)=b_m(t)e^{ikmx}\ ---(5)$$

8-2 Fourier or von Neumann analysis (8)

For an assessment of numerical stability, we are interested in the variation of with time. Therefore, we external eq.(2) by assuming the amplitude b_m is a function of time. Moreover, it is reasonable to assume an exponential variation with time; error tend to grow or diminish exponentially with time. Therefore, we write

$$\varepsilon(x,t) = e^{at} e^{ik_m x} \quad ---(6)$$

8-2 Fourier or von Neumann analysis (8)

where k_m is real, but "a" may be complex.

If eq(6) is substituted into eq(1), we obtain

$$e^{a(t+\Delta t)}e^{ik_m x} - e^{at}e^{ik_m x}$$
$$= r\left(e^{at}e^{ik_m(x+\Delta x)} - 2e^{at}e^{ik_m x} + e^{at}e^{ik_m(x-\Delta x)}\right) \ldots\ldots(7)$$

$$where \quad r = \alpha \frac{\Delta t}{(\Delta x)^2}$$

8-2 Fourier or von Neumann analysis (9)

- If we divide by $e^{at}e^{ikmx}$ and utilize the relation

$$\cos\beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$eq.(7) \quad becomes$$

$$e^{a\Delta t} = 1 + 2r(\cos\beta - 1)$$

$$where \quad \beta = k_m \Delta x, \quad Employing \quad \sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2}$$

8-2 Fourier or von Neumann analysis (10)

We can get:

$$e^{a\Delta t} = 1 - 4r\sin^2\frac{\beta}{2}$$

$$\because \frac{\varepsilon_j^{n+1}}{\varepsilon_j^n} = e^{a\Delta t}$$

8-2 Fourier or von Neumann analysis (11)

∴ the error will not grow from one time step to the next, if

$$\left|\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}\right| \le 1, \quad i.e. \quad \left|1-4r\sin^2\frac{\beta}{2}\right| \le 1$$

$$\Rightarrow -1 \le 1-4r\sin^2\frac{\beta}{2} \le 1$$

$$(1), -1 \le 1-4r\sin^2\frac{\beta}{2} \Rightarrow 4r\sin^2\frac{\beta}{2} \ge 0 \Rightarrow r \ge 0$$

$$(2), \quad 1-4r\sin^2\frac{\beta}{2} \le 1 \Rightarrow r\sin^2\frac{\beta}{2} \le \frac{1}{2} \Rightarrow r \le \frac{1}{2}$$

$$\Rightarrow (1)+(2) \Rightarrow 0 \le r \le \frac{1}{2}$$