OUTLINE

7-1 General Remarks

7-2 Finite Difference Formulation

7-3 Parabolic Equations in Two-Space Dimensions

7-1 General Remarks

- Equations of motion in fluid mechanics are frequently reduced to parabolic formulations.
- Boundary layer equations are examples of such formulations.
- In addition, the unsteady heat conduction equation is also parabolic.

7-2 Finite Difference Formulations (1)

- A typical parabolic second-order PDE is the unsteady heat conduction equation, which is considered first in one-space dimension. It has the following form

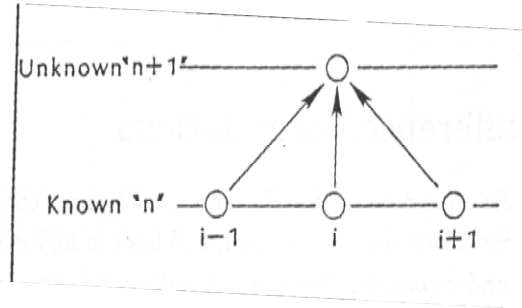
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

(1) FTCS (forward time/central space) method:

(i) $\frac{\partial u}{\partial t}$ is expressed by a forward difference approximation which is of order Δt :

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + o(\Delta t)\ldots \quad (6-1)$$

(ii) Using the second-order central differencing of order $(\Delta x)^2$ for the diffusion term , eq. (6-1) can be approximated by



$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow u_i^{n+1}=u_i^n+\frac{\alpha(\Delta t)}{(\Delta x)^2}\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)\ldots\quad(6-2)$$

(iii) Eq. (6-2) is also called explicit formulation, which is of order $\left[(\Delta t),(\Delta x)^2\right]$. It will be shown that the solution is stable for $\frac{\alpha\,\Delta t}{(\Delta x)^2}\leq\frac{1}{2}$

7-2 Finite Difference Formulations (3)

(2) BTCS (backward/central space) method:

(i)

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}$$

$$\Rightarrow \frac{\alpha\Delta t}{(\Delta x)^2}u_{i-1}^{n+1}-\left[1+2\frac{\alpha\Delta t}{(\Delta x)^2}\right]u_i^{n+1}+\frac{\alpha\Delta t}{(\Delta x)^2}u_{i+1}^{n+1}=-u_i^n \ldots \quad (6-3)$$

$$\Rightarrow a_i^n u_{i-1}^{n+1}+b_i^n u_i^{n+1}+c_i^n u_{i+1}^{n+1}=D_i^n$$

The above equation can be solved by TDMA.

7-2 Finite Difference Formulations (4)

(ii) Eq. (6-3) is defined as being implicit, since more than one unknown appears in the finite difference equation. As a result, a set of simultaneous equations needs to be solved, which require more computation time per time step. Implicit methods greater advantage on the stability of the finite difference equations, since most are unditionally stable. Therefore, a larger step size in time is permitted.

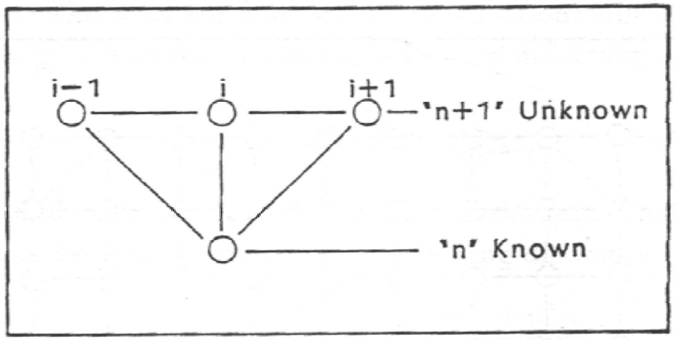


Figure 3-3. Grid points for the implicit formulation.

7-2 Finite Difference Formulations (5)

(3) CTCS (central time/central space) method: (The Crank-Nicolson method)

(i) If the diffusion term of eq. (6-1) is replaced by the average of the central differences at time levels n and n+1, the discretized equation would be of the form:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\left(\frac{1}{2}\right)\left[\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}+\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{(\Delta x)^2}\right]\ldots \quad (6-4)$$

$$\Rightarrow -ru_{i-1}^{n+1}+(2+r)u_i^{n+1}-ru_{i+1}^{n+1}=ru_{i+1}^{n}+(2-r)u_i^{n}+ru_{i-1}^{n}$$

Note: The left side of eq. (6-4) is a central difference of step, i.e., $\frac{\partial u}{\partial t}=\frac{u_i^{n+1}-u_i^n}{2(\Delta t/2)}$, which is $o(\Delta t)^2$

7-2 Finite Difference Formulations (6)

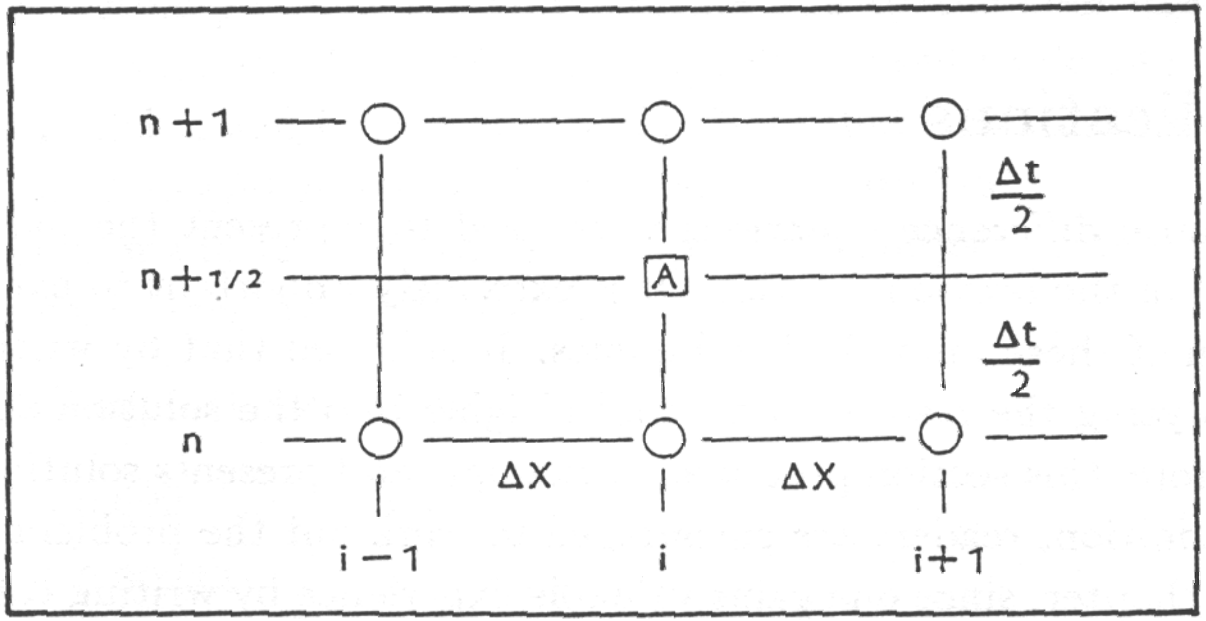


Figure 3-5. Grid points for the Crank-Nicolson implicit method

7-2 Finite Difference Formulations (7)

(ii) The method may be thought of as the addition of two step computations as follows:

Using the explicit method,

$$\frac{u_i^{n+1/2}-u_i^n}{\Delta t/2}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}\ldots\quad(6-5a)$$

while using the implicit method,

$$\frac{u_i^{n+1}-u_i^{n+1/2}}{\Delta t/2}=\alpha\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}\ldots\quad(6-5b)$$

Adding eqs. (6-5a) and (6-5b), we can get eq. (6-4).

(iii) This implicit method is unconditionally stable and is of order $\left[(\Delta t)^2,(\Delta x)^2\right]$, that is a second-order accurate scheme.

- Example

7-3 Parabolic Equations in Two-Space Dimensions (1)

- Consider the model equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

(1) FTCS (or Explicit) method:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha \left[\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right]$$

which is of order $\left[(\Delta t), (\Delta x)^2, (\Delta y)^2 \right]$

Stability analysis indicates that the method is stable for $d_x + d_y \leq 1/2$

where $d_x = \frac{\alpha \Delta t}{(\Delta x)^2}, d_y = \frac{\alpha \Delta t}{(\Delta y)^2}$

If Δx=Δy, i.e., dx=dy=d, then $d \leq 1/2$

7-3 Parabolic Equations in Two-Space Dimensions (2)

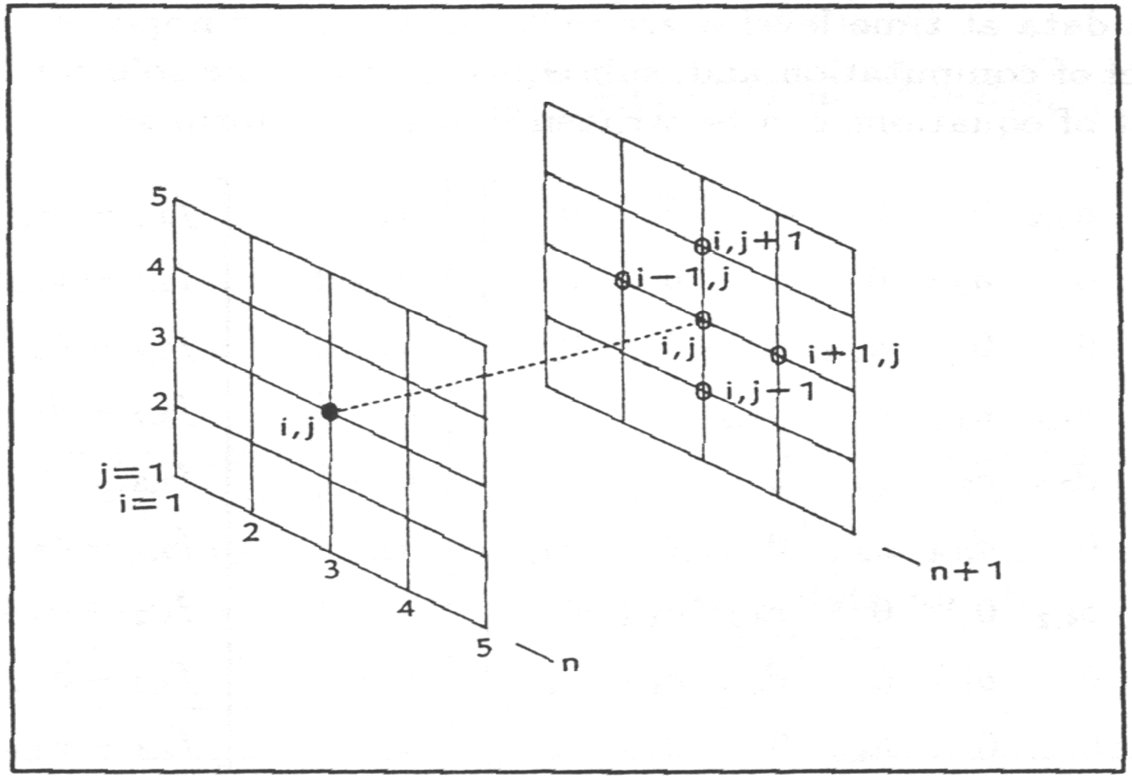


Figure 3-12. The 5 by 5 grid system for Equation (3-20).

7-3 Parabolic Equations in Two-Space Dimensions (3)

(2) Implicit (BTCS) method:

(i) Consider an implicit formulation for which the FDE is

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}=\alpha\left[\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

$$\Rightarrow d_x u_{i+1,j}^{n+1}+d_x u_{i-1,j}^{n+1}-(2d_x+2d_y+1)u_{i,j}^{n+1}+d_y u_{i,j-1}^{n+1}+d_y u_{i,j+1}^{n+1}=-u_{i,j}^{n}$$

7-3 Parabolic Equations in Two-Space Dimensions (4)

(ii) The 2-D FDEs in the ADI formulation are

$$\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{(\Delta t/2)}=\alpha\left[\frac{u_{i+1,j-}^{n+1/2}2u_{i,j}^{n+1/2}+u_{i-1,j}^{n+1/2}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

and

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n/2}}{(\Delta t/2)}=\alpha\left[\frac{u_{i+1,j-}^{n+1/2}2u_{i,j}^{n+1/2}+u_{i-1,j}^{n+1/2}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$

7-3 Parabolic Equations in Two-Space Dimensions (5)

This method is of order $\left[(\Delta t)^2, (\Delta x)^2, (\Delta y)^2\right]$ and is unconditionally stable. The above equations are written in the tridiagonal form as

$$-d_1 u_{i-1,j}^{n+1/2} + (1+2d_1)u_{i,j}^{n+1/2} - d_1 u_{i+1,j}^{n+1/2} = d_2 u_{i,j+1}^{n} + (1-2d_2)u_{i,j}^{n} + d_2 u_{i,j-1}^{n}$$

$$-d_2 u_{i,j-1}^{n+1} + (1+2d_2)u_{i,j}^{n+1} - d_2 u_{i,j+1}^{n+1} = d_1 u_{i+1,j}^{n+1/2} + (1-2d_1)u_{i,j}^{n+1/2} + d_1 u_{i-1,j}^{n+1/2}$$

where

$$d_1 = \frac{1}{2}d_x = \frac{1}{2}\frac{\alpha\Delta t}{(\Delta x)^2}, d_2 = \frac{1}{2}d_y = \frac{1}{2}\frac{\alpha\Delta t}{(\Delta y)^2}$$

7-3 Parabolic Equations in Two-Space Dimensions (6)

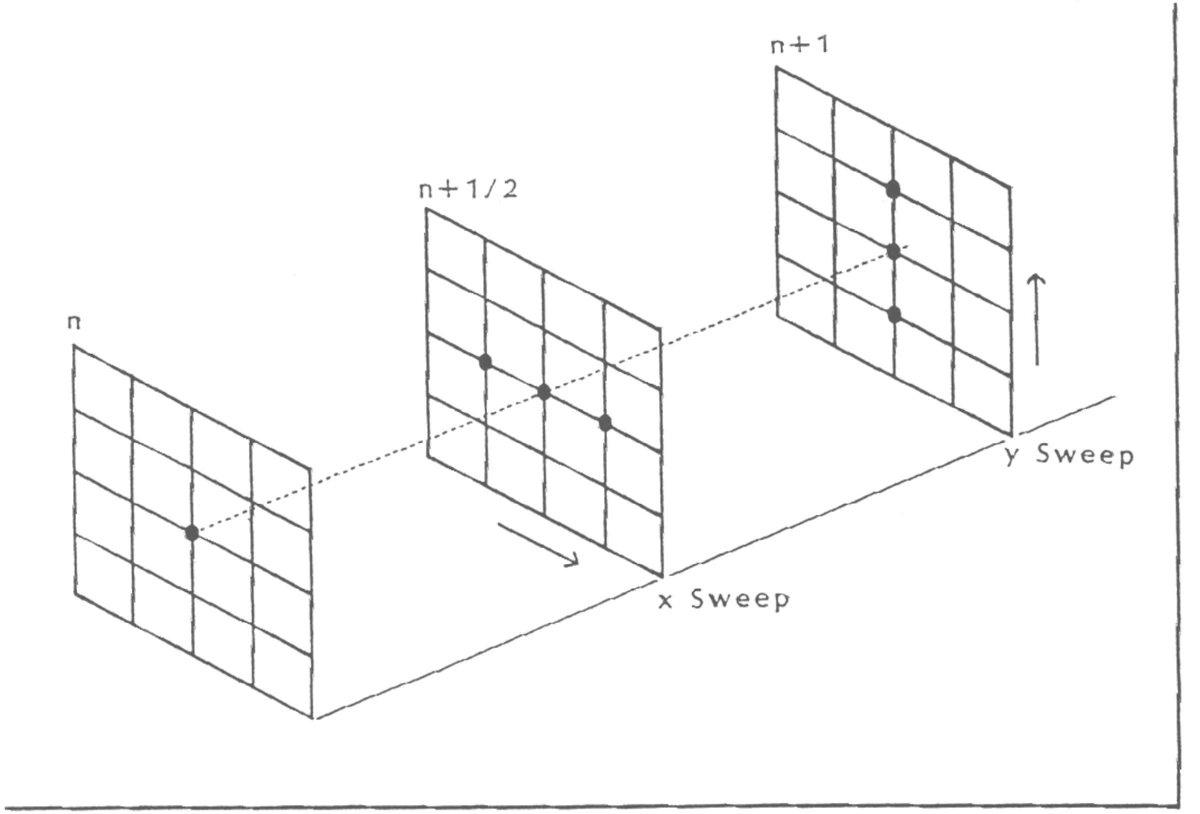


Figure 3-13. Illustration of the grid system for the ADI method.