OUTLINE

6-1 General Remarks

- The governing equations in fluid mechanics and heat transfer can be reduced to elliptic form for particular applications. Such examples are the steady-state heat conduction equation, velocity potential equation for incompressible, inviscid flow, and the stream function equation.
- Typical elliptic equations in a two-dimensional Cartesian system are Laplace's equations,

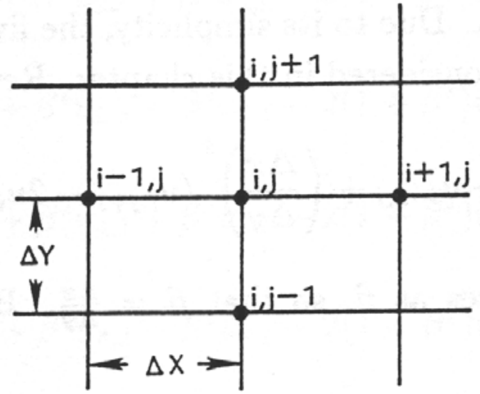
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

6-2 Finite Difference Formulations (1)

- Five-point formula---central difference



Laplace's equations: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

F.D. $$\Rightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

which is accurate to $o\left[(\Delta x)^2, (\Delta y)^2\right]$

6-2 Finite Difference Formulations (2)

- General form for five-point formula

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = 0$$

$$\Rightarrow u_{i+1,j} + u_{i-1,j} + \beta^2 \left(u_{i,j+1} + u_{i,j-1}\right) - 2\left(1 + \beta^2\right) u_{i,j} = 0 \ldots \quad (5-1)$$

$$\left(\beta = \frac{\Delta x}{\Delta y}\right)$$

$$if\ \Delta x = \Delta y \quad \Rightarrow \beta = 1$$

$$\Rightarrow u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \ldots \quad (5-2)$$

6-2 Finite Difference Formulations (3)

- From eqs. (1) and (2), we can get

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + \beta^2 \left(u_{i,j+1} + u_{i,j-1} \right)}{2\left(1+\beta^2\right)} \ldots \quad (5-3a)$$

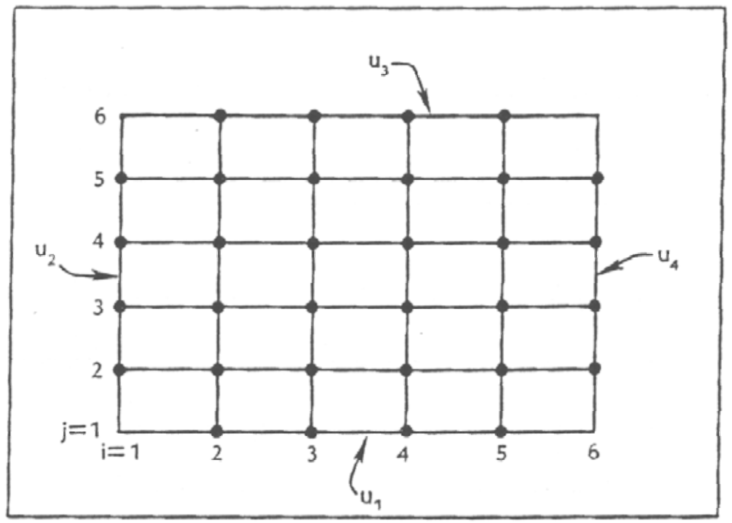
$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4} \ldots \quad (5-3b)$$

6-2 Finite Difference Formulations (4)

- In order to explore various solution procedures, first consider a square domain with Dirichlet B.C.s. For instance, a simple 6x6 grid system subject to the following B.C.s. is considered:

  $x=0 \quad u=u_2, \; y=0 \quad u=u_1$

  $x=L \quad u=u_4, \; y=H \quad u=u_3$



6-2 Finite Difference Formulations (5)

- The interior grid points produces sixteen equations with sixteen unknowns. The equations are:

$$u_{3,2}+u_{1,2}+\beta^2 u_{2,3}+\beta^2 u_{2,1}-2(1+\beta^2)u_{2,2}=0$$
$$u_{4,2}+u_{2,2}+\beta^2 u_{3,3}+\beta^2 u_{3,1}-2(1+\beta^2)u_{3,2}=0$$
$$u_{5,2}+u_{3,2}+\beta^2 u_{4,3}+\beta^2 u_{4,1}-2(1+\beta^2)u_{4,2}=0$$
$$u_{6,2}+u_{4,2}+\beta^2 u_{5,3}+\beta^2 u_{5,1}-2(1+\beta^2)u_{5,2}=0$$
$$u_{3,3}+u_{1,3}+\beta^2 u_{2,4}+\beta^2 u_{2,2}-2(1+\beta^2)u_{2,3}=0$$
$$u_{4,3}+u_{2,3}+\beta^2 u_{3,4}+\beta^2 u_{3,2}-2(1+\beta^2)u_{3,3}=0$$
$$u_{5,3}+u_{3,3}+\beta^2 u_{4,4}+\beta^2 u_{4,2}-2(1+\beta^2)u_{4,3}=0$$
$$u_{6,3}+u_{4,3}+\beta^2 u_{5,4}+\beta^2 u_{5,2}-2(1+\beta^2)u_{5,3}=0$$
$$u_{3,4}+u_{1,4}+\beta^2 u_{2,5}+\beta^2 u_{2,3}-2(1+\beta^2)u_{2,4}=0$$
$$u_{4,4}+u_{2,4}+\beta^2 u_{3,5}+\beta^2 u_{3,3}-2(1+\beta^2)u_{3,4}=0$$
$$u_{5,4}+u_{3,4}+\beta^2 u_{4,5}+\beta^2 u_{4,3}-2(1+\beta^2)u_{4,4}=0$$
$$u_{6,4}+u_{4,4}+\beta^2 u_{5,5}+\beta^2 u_{5,3}-2(1+\beta^2)u_{5,4}=0$$
$$u_{3,5}+u_{1,5}+\beta^2 u_{2,6}+\beta^2 u_{2,4}-2(1+\beta^2)u_{2,5}=0$$
$$u_{4,5}+u_{2,5}+\beta^2 u_{3,6}+\beta^2 u_{3,4}-2(1+\beta^2)u_{3,5}=0$$
$$u_{5,5}+u_{3,5}+\beta^2 u_{4,6}+\beta^2 u_{4,4}-2(1+\beta^2)u_{4,5}=0$$
$$u_{6,5}+u_{4,5}+\beta^2 u_{5,6}+\beta^2 u_{5,4}-2(1+\beta^2)u_{5,5}=0$$

$$\begin{bmatrix}
\alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & \alpha & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\beta^2 & 0 & 0 & 0 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \beta^2 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & \beta^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & \beta^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & \beta^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & \alpha & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 1 & \alpha
\end{bmatrix}
\begin{bmatrix} u_{2,2}\\ u_{3,2}\\ u_{4,2}\\ u_{5,2}\\ u_{2,3}\\ u_{3,3}\\ u_{4,3}\\ u_{5,3}\\ u_{2,4}\\ u_{3,4}\\ u_{4,4}\\ u_{5,4}\\ u_{2,5}\\ u_{3,5}\\ u_{4,5}\\ u_{5,5} \end{bmatrix}
=
\begin{bmatrix} -u_{1,2}-\beta^2 u_{2,1}\\ -\beta^2 u_{3,1}\\ -\beta^2 u_{4,1}\\ -u_{6,2}-\beta^2 u_{5,1}\\ -u_{1,3}\\ 0\\ 0\\ -u_{6,3}\\ -u_{1,4}\\ 0\\ 0\\ -u_{6,4}\\ -u_{1,5}-\beta^2 u_{3,6}\\ -\beta^2 u_{3,6}\\ -\beta^2 u_{4,6}\\ -u_{6,5}-\beta^2 u_{5,6} \end{bmatrix}
\tag{5-7}$$

where $\alpha=-2(1+\beta^2)$.

The matrix formulation has two noteworthy features. First, it is a pentadiagonal matrix with nonadjacent diagonals; and second, the elements in the main diagonal in each row are the largest. These features are important when developing solution procedures.

6-2 Finite Difference Formulations (6)

- Solution algorithms:

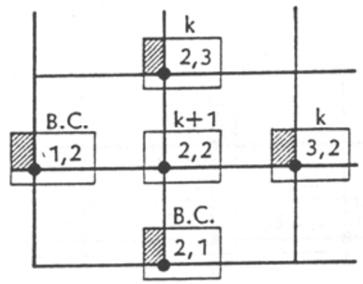
  (1) The Gauss-Seidel Iteration Method (point-by-point iteration method):

  (a) The finite difference equation is given by

$$u_{i,j} = \frac{1}{2\left(1+\beta^2\right)}\left[u_{i+1,j} + u_{i-1,j} + \beta^2\left(u_{i,j+1} + u_{i,j-1}\right)\right]$$

  (b) For the computation of the first point, say (2,2), it follows that

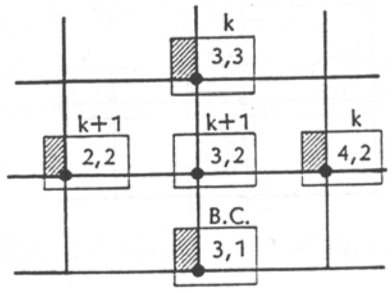
$$u_{2,2}^{k+1} = \frac{1}{2\left(1+\beta^2\right)}\left[u_{3,2}^{k} + \underline{u_{1,2}} + \beta^2\left(u_{2,3}^{k} + \underline{u_{2,1}}\right)\right]$$



6-2 Finite Difference Formulations (7)

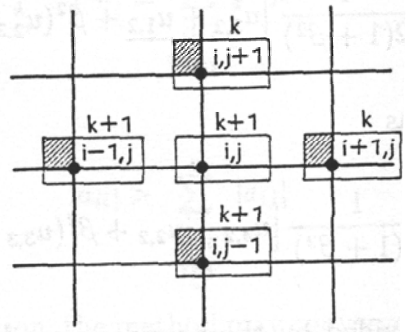
(c) For point (3,2), one has

$$u_{3,2}^{k+1}=\frac{1}{2\left(1+\beta^2\right)}\left[u_{4,2}^{k}+u_{2,2}^{k+1}+\beta^2\left(u_{2,3}^{k}+\underline{u_{3,1}}\right)\right]$$



(d) The general formulation is

$$u_{i,j}^{k+1}=\frac{1}{2\left(1+\beta^2\right)}\left[u_{i+1,j}^{k}+u_{i-1,j}^{k+1}+\beta^2\left(u_{i,j+1}^{k}+u_{i,j-1}^{k+1}\right)\right]$$



(2) SOR Method:

$$u_{i,j}^{k+1} = u_{i,j}^{k} + \frac{\omega}{2\left(1+\beta^2\right)}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k+1} + \beta^2\left(u_{i,j+1}^{k} + u_{i,j-1}^{k+1}\right) - 2\left(1+\beta^2\right)u_{i,j}^{k}\right]$$

or

$$u_{i,j}^{k+1} = \left(1-\omega\right)u_{i,j}^{k} + \frac{\omega}{2\left(1+\beta^2\right)}\left[u_{i+1,j}^{k} + u_{i-1,j}^{k+1} + \beta^2\left(u_{i,j+1}^{k} + u_{i,j-1}^{k+1}\right)\right]$$

What is the optimum value of ω?

One such relation, for the solution of elliptic equations in a rectangular domain subject to Dirichlet B.C.s with constant step size, is

$$\omega_{opt} = \frac{2-2\sqrt{1-a}}{a}, \qquad a = \left[\frac{\cos\left(\dfrac{\pi}{IM-1}\right) + \beta^2\cos\left(\dfrac{\pi}{JM-1}\right)}{1+\beta^2}\right]^2$$

6-2 Finite Difference Formulations (9)

(3) Line-by-line iteration method:

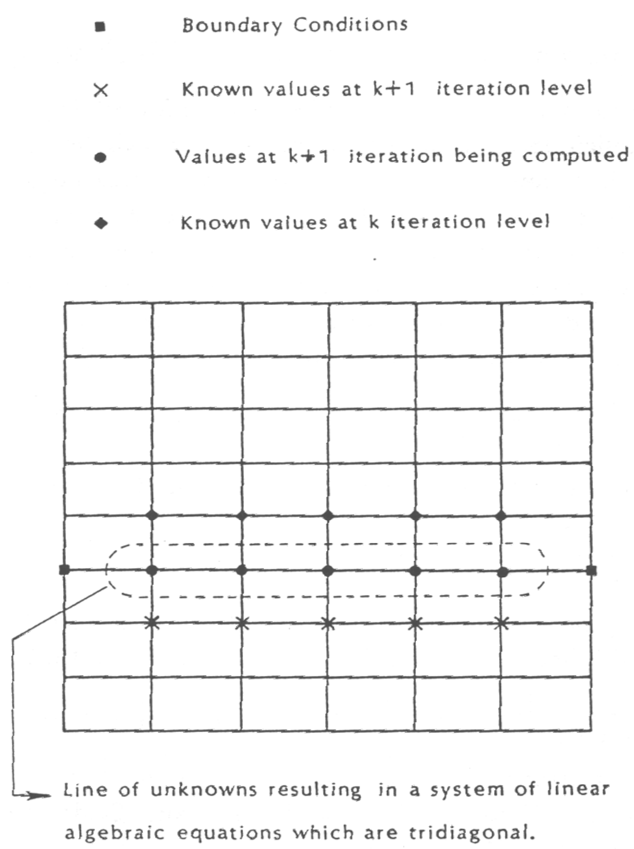
(a) In this formulation, it results in three unknowns at points (i-1,j), (i,j), and (i+1,j). It becomes

$$u_{i-1,j}^{k+1} - 2\left(1+\beta^2\right)u_{i,j}^{k+1} + u_{i+1,j}^{k+1} = -\beta^2 u_{i,j+1}^{k} - \beta^2 u_{i,j-1}^{k+1}$$

(b) This equation, applied to all i at constant j, results in a system of linear equations which, in compact form, has a tridiagonal matrix coefficient. The solution of each row at constant j can be solved by TDMA method.

6-2 Finite Difference Formulations (10)

(c) Grid points employed in the line-by line iteration method.



6-2 Finite Difference Formulations (11)

(4) Line-by line SOR method:

$$\omega u_{i-1,j}^{k+1} - 2\left(1+\beta^2\right)u_{i,j}^{k+1} + \omega u_{i+1,j}^{k+1} =$$
$$-\left(1-\omega\right)\left[2\left(1+\beta^2\right)\right]u_{i,j+1}^{k} - \omega\beta^2\left(u_{i,j+1}^{k} u_{i,j-1}^{k+1}\right)$$

There is no simple way to determine the value of optimum ω.

In practice, trial and error is used to compute ω_{opt} for a particular problem.

(5) The Alternating Direction Implicit (ADI) Method:

(a) An iteration cycle is considered complete once the resulting tridiagonal system is solved for all rows and then followed by columns, or vice versa. It follows

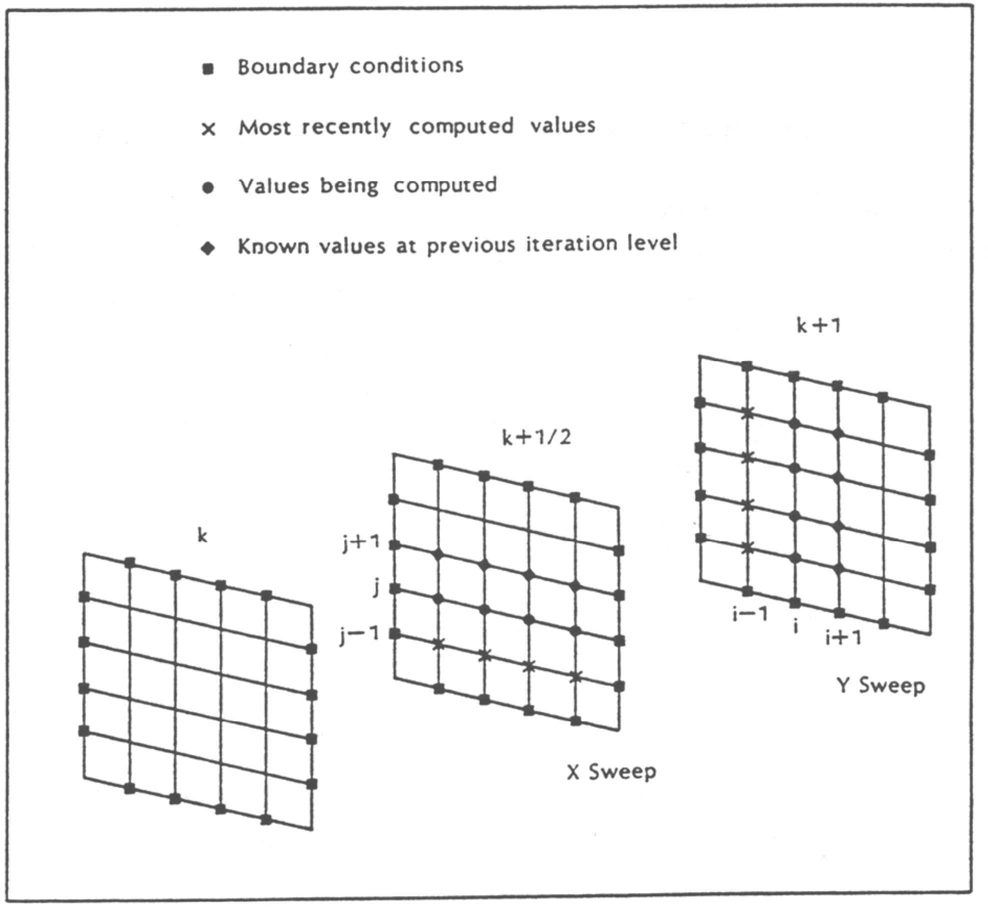
$$u_{i-1,j}^{k+\frac{1}{2}} - 2\left(1+\beta^2\right)u_{i,j}^{k+\frac{1}{2}} + u_{i+1,j}^{k+\frac{1}{2}} = -\beta^2 u_{i,j+1}^{k} - \beta^2 u_{i,j-1}^{k+\frac{1}{2}}$$

and

$$\beta^2 u_{i,j-1}^{k+1} - 2\left(1+\beta^2\right)u_{i,j}^{k+1} + u_{i,j+1}^{k+1} = -u_{i+1,j}^{k+\frac{1}{2}} - u_{i-1,j}^{k+1}$$

6-2 Finite Difference Formulations (13)

(b) Grid points used in ADI method:



6-2 Finite Difference Formulations (14)

(c) ADI with SOR Method:

$$\omega u_{i-1,j}^{k+\frac{1}{2}} - 2\left(1+\beta^2\right)u_{i,j}^{k+\frac{1}{2}} + \omega u_{i+1,j}^{k+\frac{1}{2}} =$$

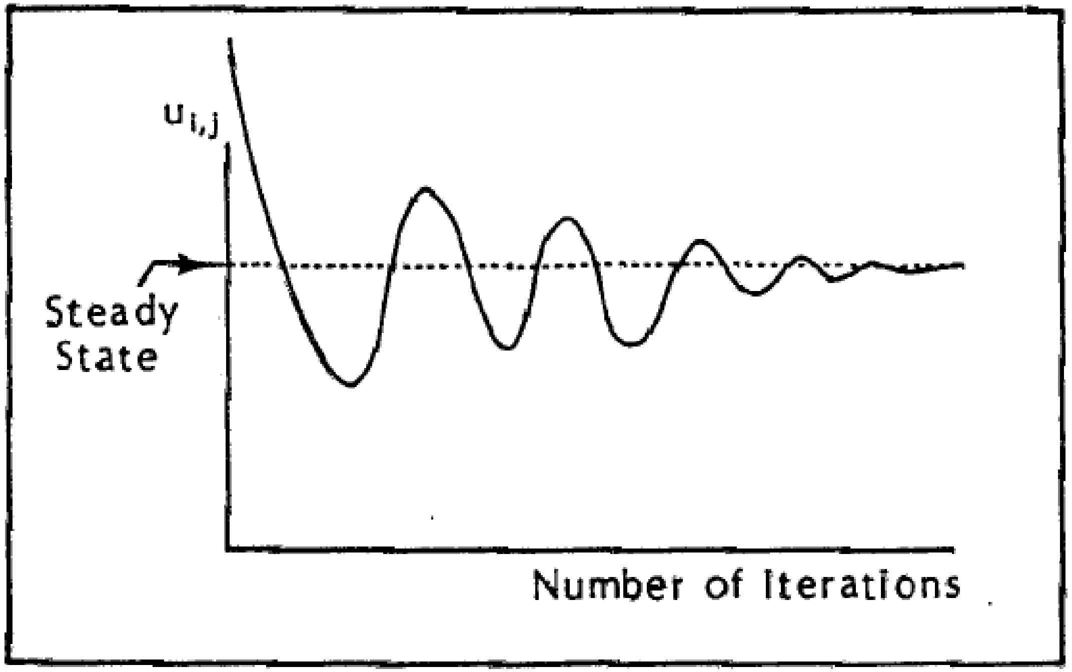
$$-\left(1-\omega\right)\left[2\left(1+\beta^2\right)\right]u_{i,j}^{k} - \omega\beta^2\left(u_{i,j+1}^{k} + u_{i,j-1}^{k+\frac{1}{2}}\right)$$

and

$$\omega\beta^2 u_{i,j-1}^{k+1} - 2\left(1+\beta^2\right)u_{i,j}^{k+1} + \omega\beta^2 u_{i,j+1}^{k+1} =$$

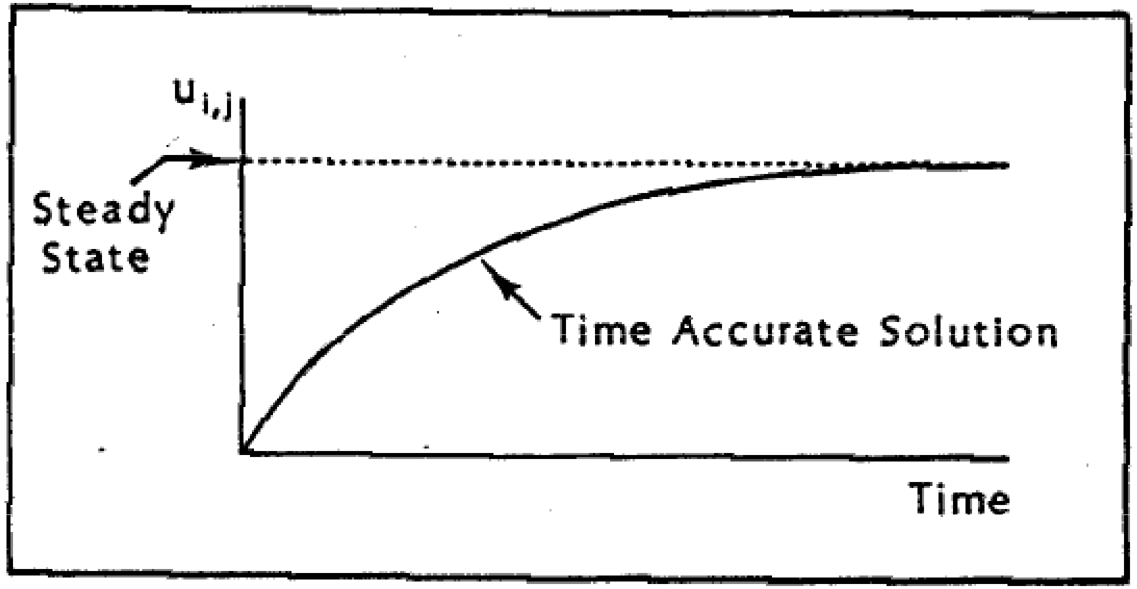
$$-\left(1-\omega\right)\left[2\left(1+\beta^2\right)\right]u_{i,j}^{k+\frac{1}{2}} - \omega\left(u_{i+1,j}^{k+\frac{1}{2}} + u_{i-1,j}^{k+1}\right)$$

6-2 Finite Difference Formulations (15)



An iterative solution for steady-state problem

6-2 Finite Difference Formulations (16)



A time accurate solution