Chapter 2: Basic Equations in Integral Form for a Control Volume

2-1 Reynolds Transport Theorem (RTT)

2-2 Continuity Equation

2-3 The Linear Momentum Equation

2-4 The First Law of Thermodynamics

2-5 The Second Law of Thermodynamics

2-1 Reynolds Transport Theorem (RTT) (1)

■ Conservation of Mass:

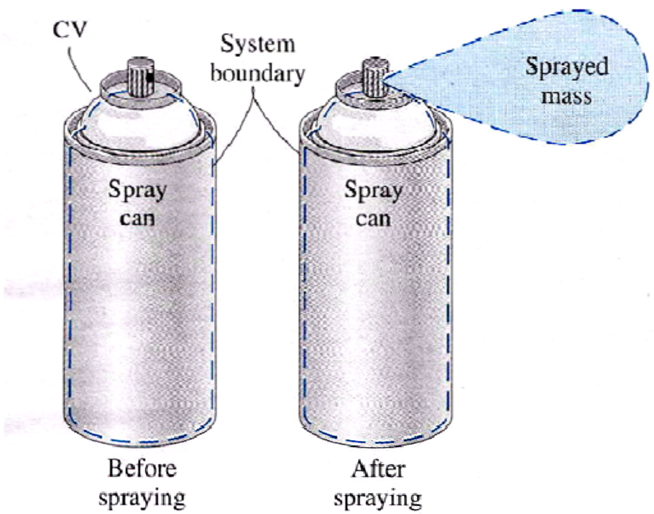


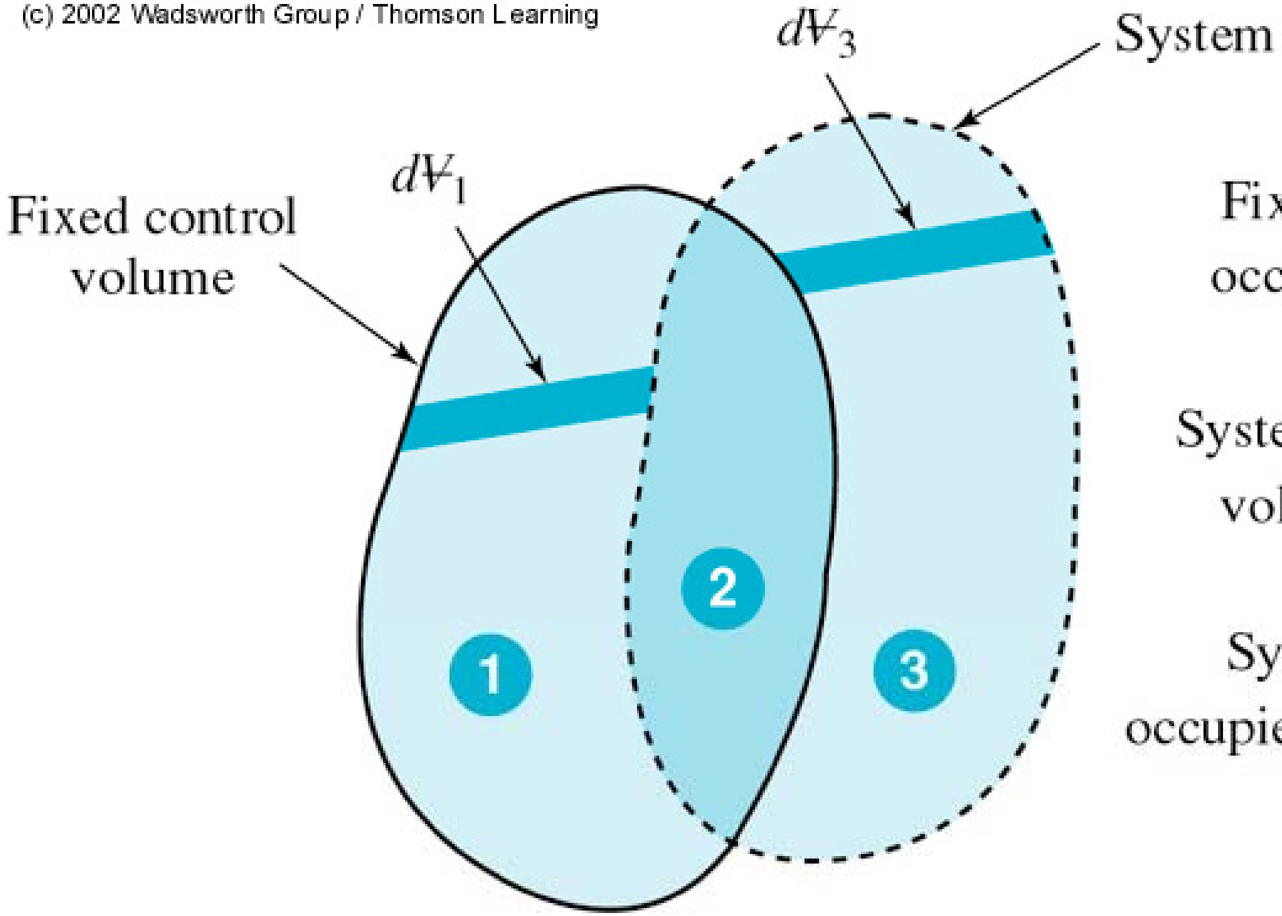
FIGURE 11-38

A spray can can be analyzed by taking either the contents of a can (a closed system) or the interior volume of the can (a control volume) as the system.

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0$$

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\forall(\text{system})} \rho \, d\forall$$

2-1 Reynolds Transport Theorem (RTT) (2)



Fixed control volume occupies 1 and 2.

System at time t occupies volumes 1 and 2.

System at time $t + \Delta t$ occupies volumes 2 and 3.

System and fixed control volume.

2-1 Reynolds Transport Theorem (RTT) (3)

$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b(\vec{V} \cdot \hat{n})\, dA \quad \text{(inflow if negative)} \quad (11\text{-}41)$$

$$B_{CV} = \int_{CV} \rho b\, dV \qquad (11\text{-}42)$$

$$General: \quad \frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b(\vec{V} \cdot \hat{n})\, dA \qquad (11\text{-}43)$$

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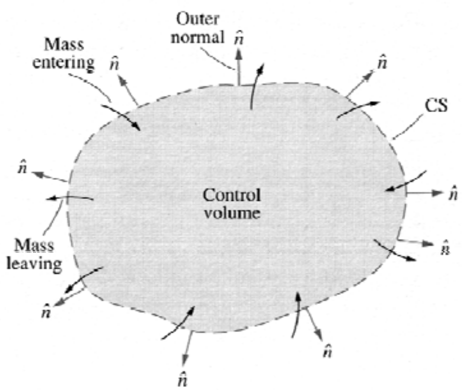
CHAPTER 11
Bernoulli, Energy, and Momentum Equations



$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b(\vec{V} \cdot \hat{n})\, dA$$

FIGURE 11-40
The integral of $b\rho(\vec{V} \cdot \hat{n})\, dA$ over the control surface gives the net amount of the property b flowing out of the control volume (into the control volume if it is negative), per unit time.

■ Interpreting the Scalar Product:

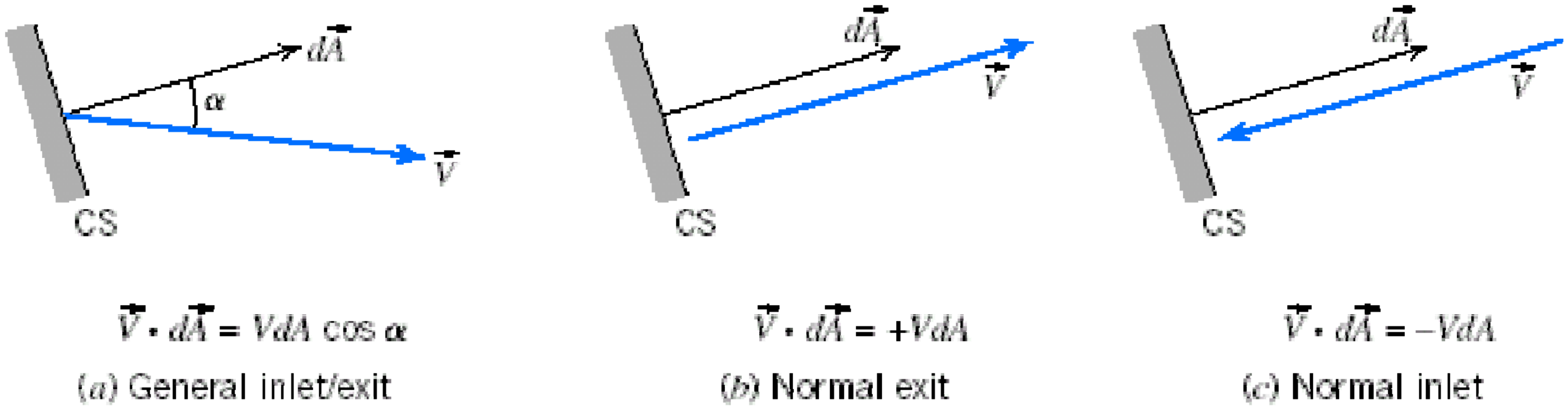


Fig. 4.3 Evaluating the scalar product.

2-1 Reynolds Transport Theorem (RTT) (5)

❑ Special Case 1: Steady Flow

$$Steady\ flow: \qquad \frac{dB_{sys}}{dt} = \int_{\mathrm{CS}} \rho b(\vec{V} \cdot \hat{n})dA \qquad (11\text{-}44)$$

❑ Special Case 2: One-Dimensional Flow

$$One\text{-}dimensional\ flow:$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b\, dV + \sum_{\mathrm{out}} \underbrace{\rho_e b_e V_e A_e}_{\text{for each exit}} - \sum_{\mathrm{in}} \underbrace{\rho_i b_i V_i A_i}_{\text{for each exit}} \qquad (11\text{-}45)$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b\, dV + \sum_{\mathrm{out}} \dot{m}_e b_e - \sum_{\mathrm{in}} \overset{*}{m}_i b_i \qquad (11\text{-}46)$$

2-1 Reynolds Transport Theorem (RTT) (6)

■ Extensive and Intensive Properties:

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dm = \int_{\forall(\text{system})} \eta \, \rho \, d\forall$$

$$N = M, \eta = 1$$

$$N = \vec{P}, \quad \eta = \vec{V}$$

$$N = \vec{H}, \eta = \vec{r} \times \vec{V}$$

$$N = E, \quad \eta = e$$

$$N = S, \quad \eta = s$$

2-2 Continuity Equation (1)

❑ An Application: The Continuity Equation

$$\text{Continuity equation: } \quad 0 = \frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho(\vec{V}\cdot\hat{n})\, dA \qquad (11\text{-}47)$$

$$\text{Steady flow: } \quad \sum_{out} \dot{m}_e = \sum_{in} \dot{m}_i \qquad (11\text{-}48)$$

$$\text{Single stream}, \rho = \text{constant}:$$

$$\rho V_1 A_1 = \rho V_2 A_2 \text{ or } V_1 A_1 = V_2 A_2 \qquad (11\text{-}49)$$

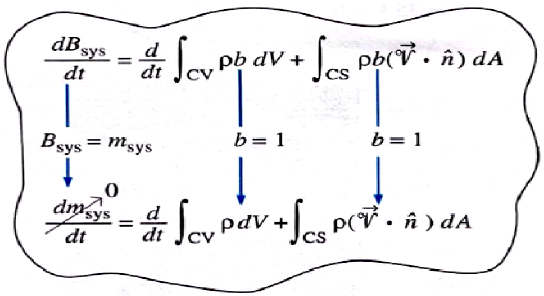


FIGURE 11-43
The continuity equation is obtained by replacing B in the Reynolds transport theorem by the mass m and b by 1 (m per unit mass $= m/m = 1$).

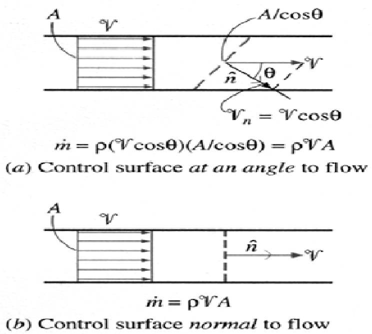
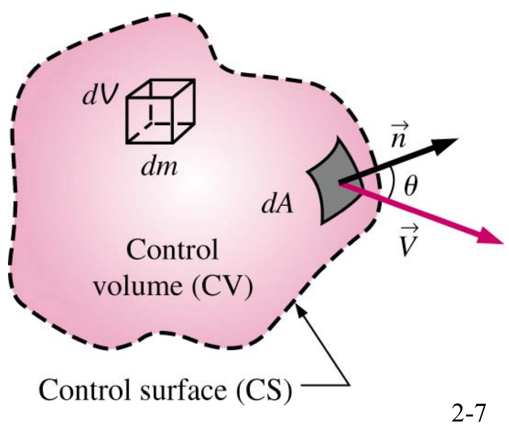


FIGURE 11-44
A control surface should always be selected *normal to flow* at all locations it crosses fluid flow to avoid complications.



2-2 Continuity Equation (2)

Conservation of Mass:

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall = -\int_{CS}\rho\vec{V}\cdot d\vec{A}$$

Rate of increase of mass in CV = Net influx of mass

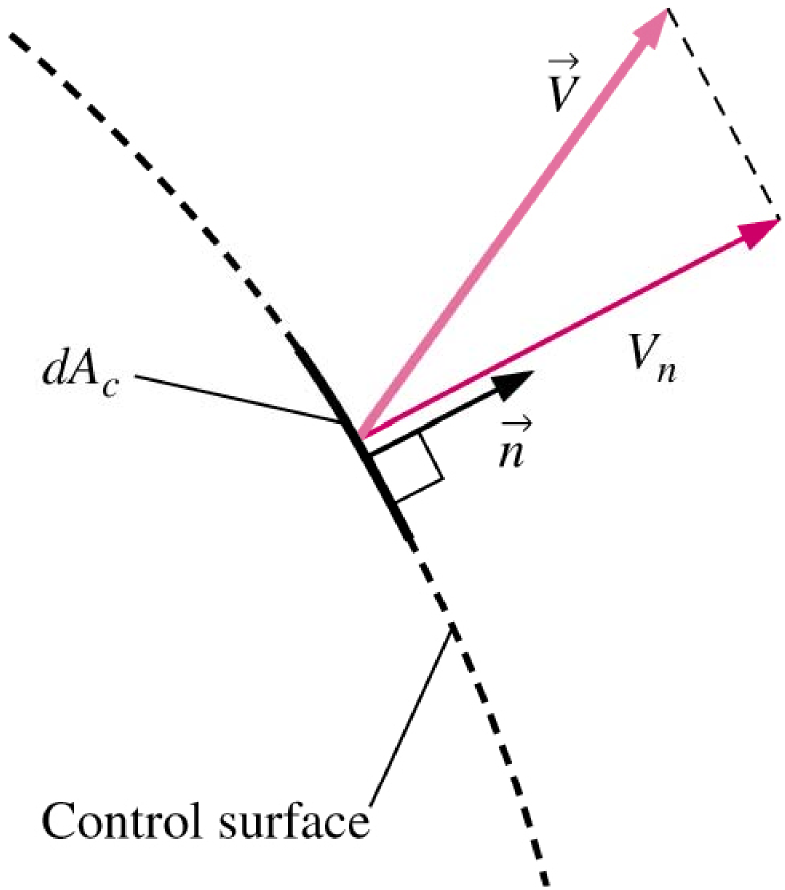
- ✓ Incompressible Fluids

$$\int_{CS}\vec{V}\cdot d\vec{A} = 0$$

- ✓ **Steady, Compressible Flow**

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

2-2 Continuity Equation (3)

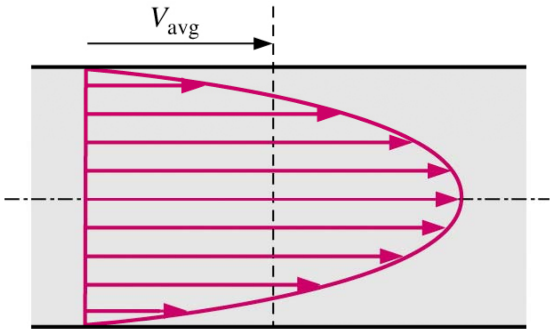


- The amount of mass flowing through a control surface per unit time is called the **mass flow rate** and is denoted $\dot{m}$
- The dot over a symbol is used to indicate *time rate of change*.
- Flow rate across the entire cross-sectional area of a pipe or duct is obtained by integration

$$\dot{m} = \int_{A_c} \delta m = \int_{A_c} \rho V_n dA_c$$

- While this expression for $\dot{m}$ is exact, it is not always convenient for engineering analyses.

Average Velocity and Volume Flow Rate:



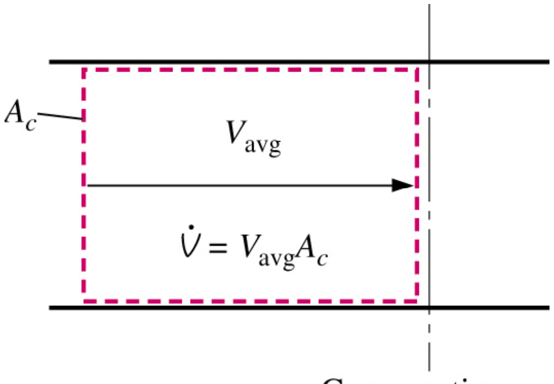
✓Integral in $\dot{m}$ can be replaced with average values of ρ and V_n

$$V_{avg} = \frac{1}{A_c}\int_{A_c} V_n dA_c$$

✓For many flows variation of r is very small: $\dot{m} = \rho V_{avg} A_c$

✓Volume flow rate $\dot{V}$ is given by

$$\dot{V} = \int_{A_c} V_n dA_c = V_{avg} A_c = V A_c$$



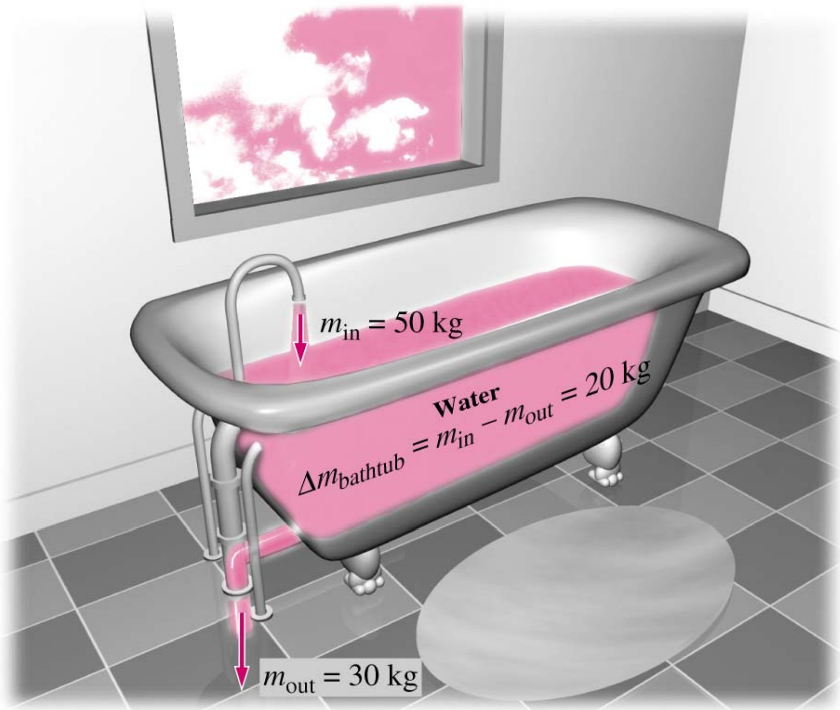
✓Note: many textbooks use Q instead of $\dot{V}$ for volume flow rate.

✓Mass and volume flow rates are related by

$$\dot{m} = \rho \dot{V}$$

2-2 Continuity Equation (5)

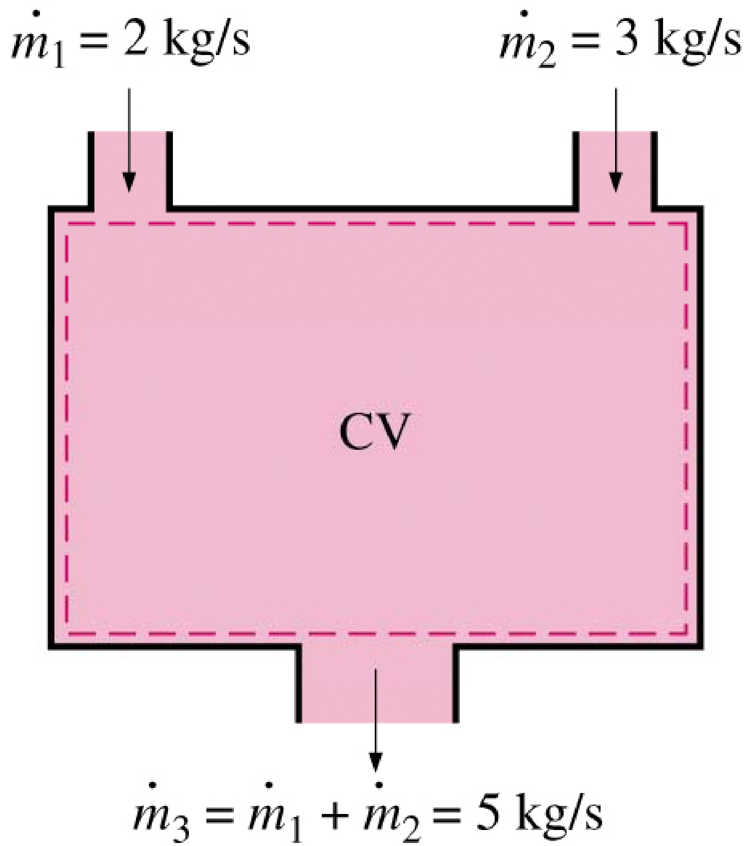
Conservation of Mass Principle:



- The **conservation of mass principle** can be expressed as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

Steady—Flow Processes:



✓For steady flow, the total amount of mass contained in CV is constant.

✓Total amount of mass entering must be equal to total amount of mass leaving

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

✓For incompressible flows,

$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

2-3 The Linear Momentum Equation (1)

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V})$$

$$\sum \vec{F} = \frac{d}{dt}\int_{sys} \vec{V}\rho dV \qquad (11\text{-}51)$$

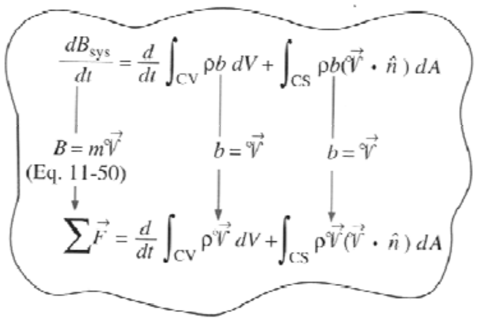


FIGURE 11-46
The linear momentum equation is obtained by replacing B in the Reynolds transport theorem by the momentum $m\vec{V}$ and b by the velocity $\vec{V}$.

$$\frac{d(m\vec{V})_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\vec{V} dV + \int_{CS} \rho\vec{V}(\vec{V}\cdot\hat{n})dA \qquad (11\text{-}52)$$

$$General: \quad \sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V} dV + \int_{CS} \rho\vec{V}(\vec{V}\cdot\hat{n})dA \qquad (11\text{-}53)$$

2-3 The Linear Momentum Equation (2)

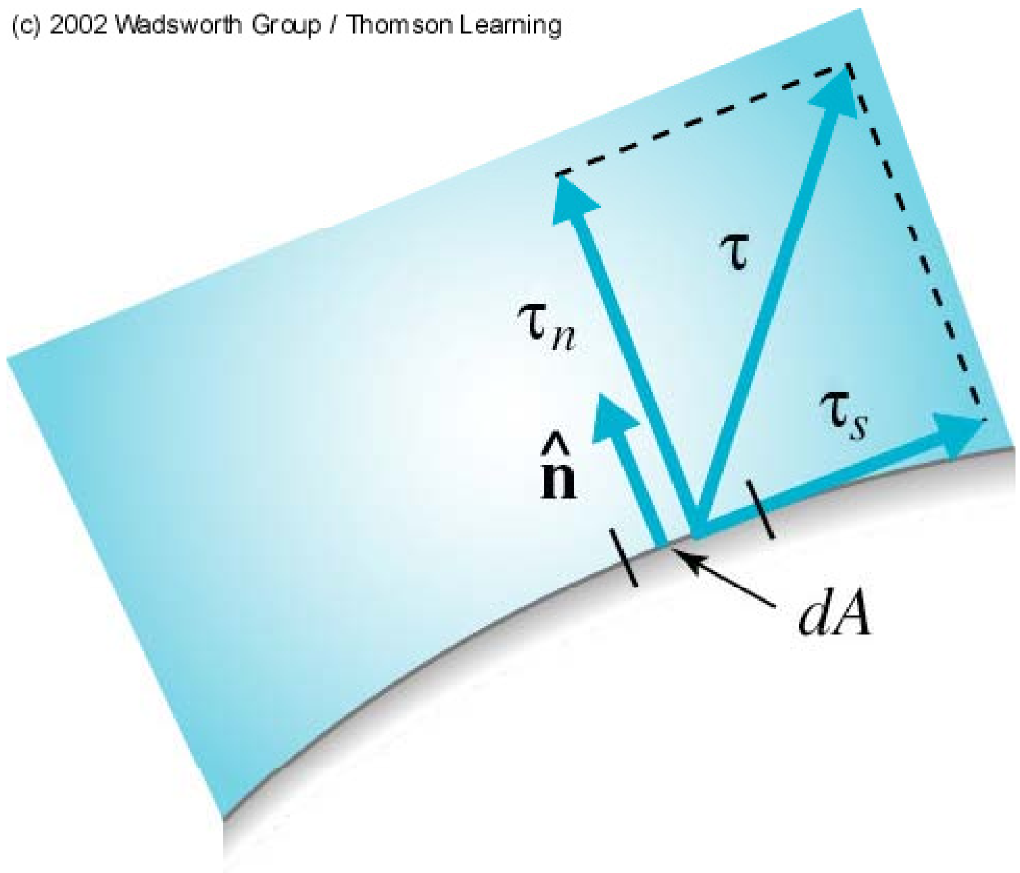
$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}\,\rho\, d\forall + \int_{\mathrm{CS}} \vec{V}\,\rho \vec{V} \cdot d\vec{A}$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u\,\rho\, d\forall + \int_{\mathrm{CS}} u\,\rho \vec{V} \cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} v\,\rho\, d\forall + \int_{\mathrm{CS}} v\,\rho \vec{V} \cdot d\vec{A}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} w\,\rho\, d\forall + \int_{\mathrm{CS}} w\,\rho \vec{V} \cdot d\vec{A}$$

2-3 The Linear Momentum Equation (3)



Stress vector acting on the control surface.

2-3 The Linear Momentum Equation (4)

❑ Special Cases

Steady flow:

$$\sum \vec{F} = \int_{CS} \rho \vec{V}(\vec{V} \cdot \hat{n}) dA \qquad (11\text{-}54)$$

One-dimensional flow:

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{out} \dot{m}_e \vec{V}_e - \sum_{in} \dot{m}_i \vec{V}_i \qquad (11\text{-}55)$$

Steady, one-dimensional flow:

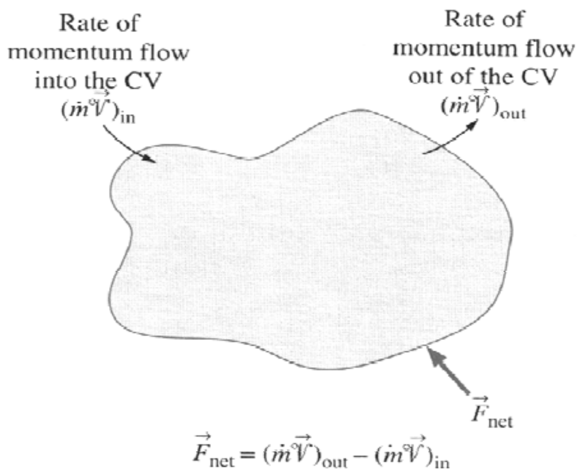
$$\sum \vec{F} = \sum_{out} \dot{m}_e \vec{V}_e - \sum_{in} \dot{m}_i \vec{V}_i \qquad (11\text{-}56)$$

2-3 The Linear Momentum Equation (5)

Steady, one-dimensional flow

(one-inlet, one-exit): $$\sum \vec{F} = \dot{m}(\vec{V}_2 - \vec{V}_1) \qquad (11\text{-}57)$$

Along x coordinate: $$\sum \vec{F}_x = \dot{m}(\vec{V}_{2,x} - \vec{V}_{1,x}) \qquad (11\text{-}58)$$



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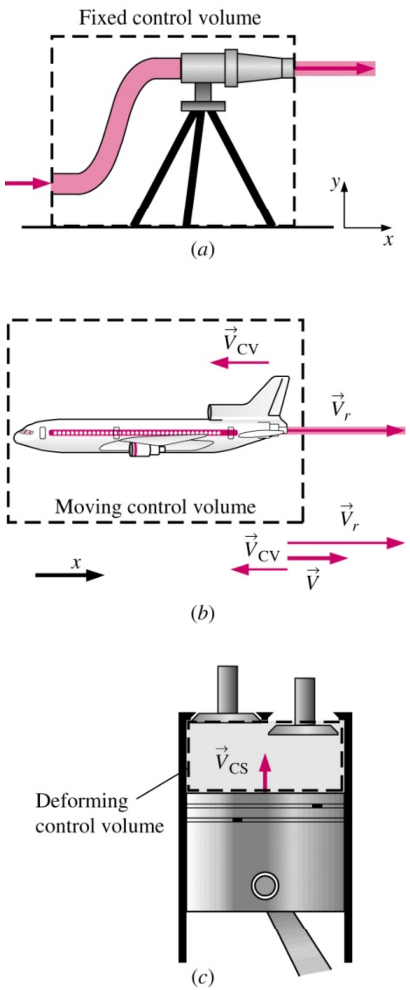
The Linear Momentum Equation

FIGURE 11-48

The net force acting on the control volume during steady flow is equal to the difference between the outgoing and the incoming momentum fluxes by mass.

2-3 The Linear Momentum Equation (6)

Choosing a Control Volume:



- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
  - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
  - Clearly identify all fluxes crossing the CS.
  - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.
  - For moving CV, use relative velocity,

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

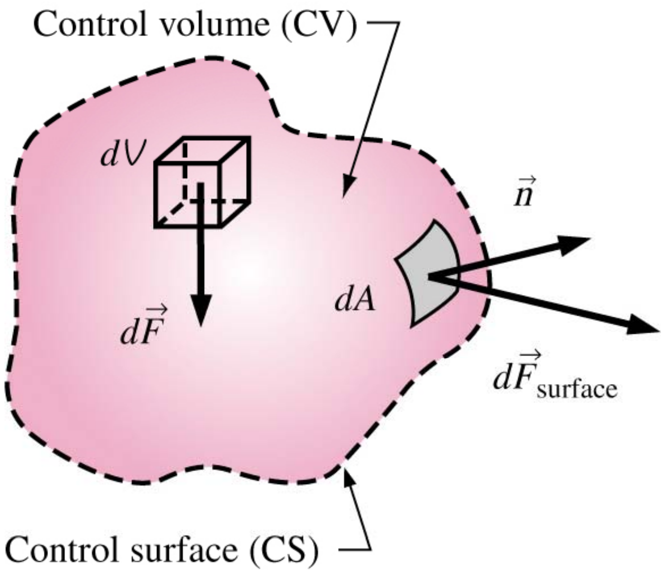
  - For deforming CV, use relative velocity all deforming control surfaces,

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

2-3 The Linear Momentum Equation (7)

Forces Acting on a CV:

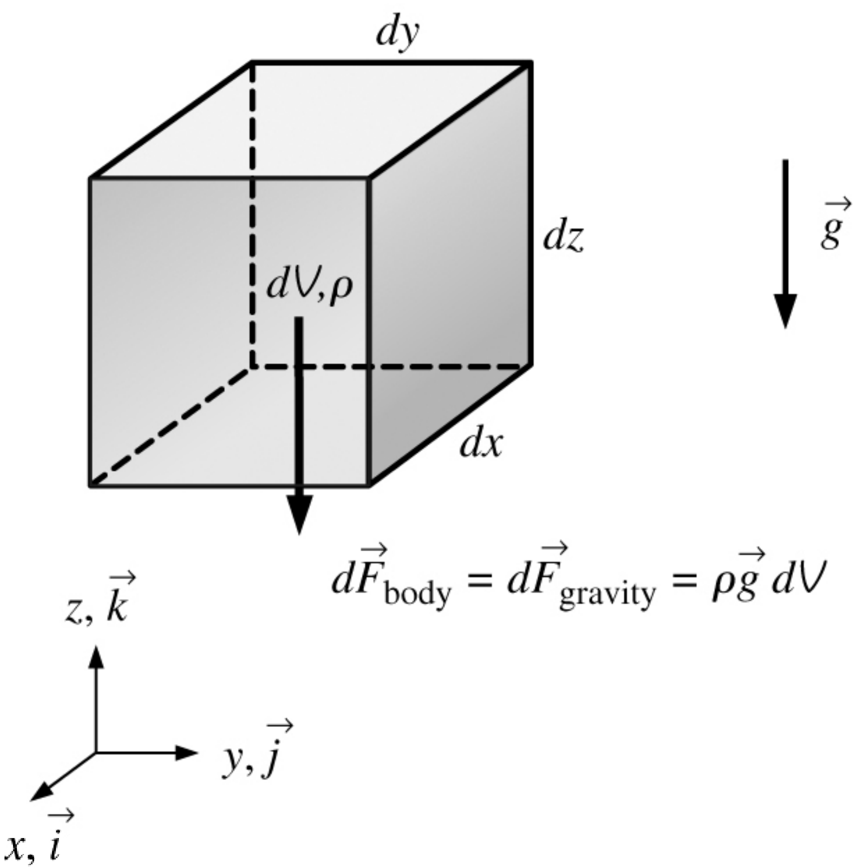
- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion dV of the CV.
- Surface forces act on each portion dA of the CS.

2-3 The Linear Momentum Equation (8)

- **Body Forces:**



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The different body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

- Typical convention is that acts in the negative z-direction,

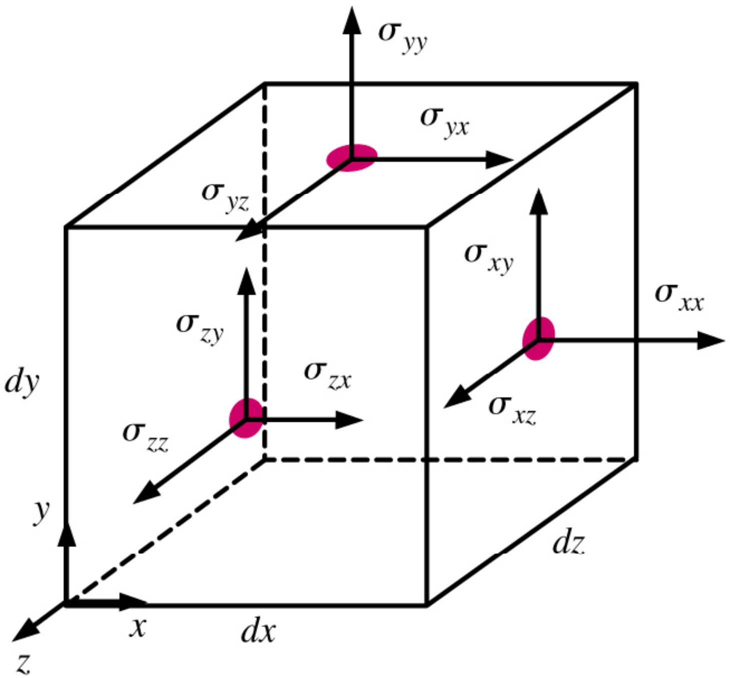
$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

2-3 The Linear Momentum Equation (9)

- **Surface Forces:**



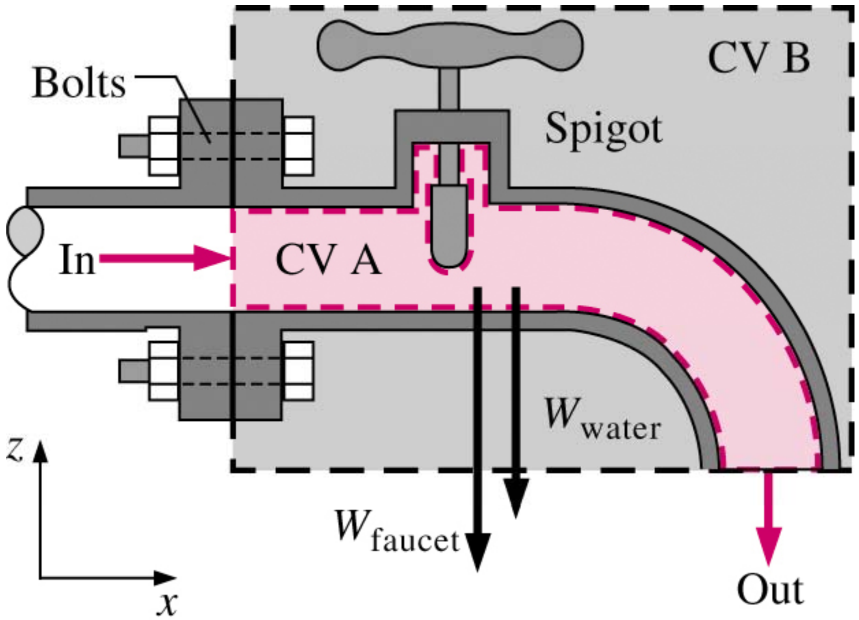
- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components σ_{xx}, σ_{yy}, σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy}, σ_{xz}, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

2-3 The Linear Momentum Equation (10)

Body and Surface Forces



- Surface integrals are cumbersome.
- Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, pressure, and reaction forces.
- Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

2-3 The Linear Momentum Equation (11)

- **Special Case: Control Volume Moving with Constant Velocity:**

$$\vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}_{xyz}\,\rho\, d\forall + \int_{\mathrm{CS}} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Momentum Equation for Inertial Control Volume with Rectilinear Acceleration

$$\vec{F}_S + \vec{F}_B - \int_{\mathrm{CV}} \vec{a}_{rf}\,\rho\, d\forall = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}_{xyz}\,\rho\, d\forall + \int_{\mathrm{CS}} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

2-4 The First Law of Thermodynamics (1)

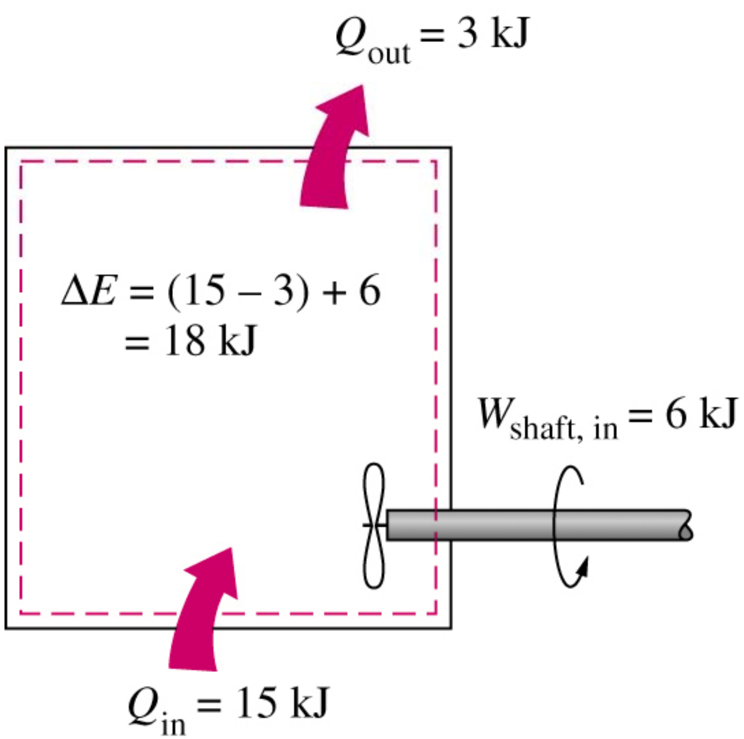
■ Basic Law, and Transport Theorem:

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{\text{system}}$$

$$\left.\frac{dE}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} e\,\rho\vec{V}\cdot d\vec{A}$$

2-4 The First Law of Thermodynamics (2)

General Energy Equation:



- The energy content of a closed system can be changed by two mechanisms: *heat transfer Q* and *work transfer W*.
- Conservation of energy for a closed system can be expressed in rate form as

$$\dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{dE_{sys}}{dt}$$

- Net rate of heat transfer to the system:

$$\dot{Q}_{net,in} = \dot{Q}_{in} - \dot{Q}_{out}$$

- Net power input to the system:

$$\dot{W}_{net,in} = \dot{W}_{in} - \dot{W}_{out}$$

2-4 The First Law of Thermodynamics (3)

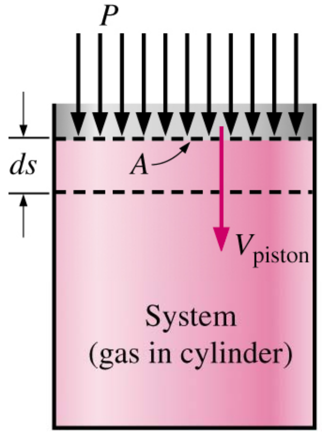
$b = e$ $\qquad\qquad e = u + \frac{V^2}{2} + gz$

$$\dot{Q} - \dot{W} = \frac{dE}{dt}\Big)_{system} = \frac{\partial}{\partial t}\int_{cv} e\rho dV + \int_{cs} e\rho \vec{V} \bullet d\vec{A}$$

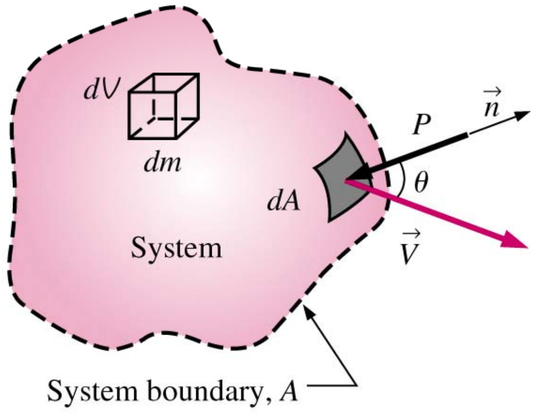
$$e = u + \frac{V^2}{2} + gz$$

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{normal} + \dot{W}_{shear} + \dot{W}_{other}$$

2-4 The First Law of Thermodynamics (4)



(*a*)



(*b*)

- Where does expression for pressure work come from?
- When piston moves down *ds* under the influence of *F=PA*, the work done on the system is $\delta W_{boundary} = PAds$.
- If we divide both sides by *dt*, we have

$$\delta \dot{W}_{pressure} = \delta \dot{W}_{boundary} = PA\frac{ds}{dt} = PAV_{piston}$$

- For generalized control volumes:

$$\delta \dot{W}_{pressure} = -PdAV_n = -PdA\left(\vec{V} \cdot \vec{n}\right)$$

- Note sign conventions:
  - $\vec{n}$ is outward pointing normal
  - Negative sign ensures that work done is positive when is done *on* the system.

2-4 The First Law of Thermodynamics (5)

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} =$$

$$\frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} \left(u + pv + \frac{V^2}{2} + gz \right) \rho \vec{V} \cdot d\vec{A}$$

Work Involves

- ✓ **Shaft Work**
- ✓ **Work by Shear Stresses at the Control Surface**
- ✓ **Other Work**

Recall that $P\upsilon$ is the **flow work,** which is the work associated with pushing a fluid into or out of a CV per unit mass.

2-4 The First Law of Thermodynamics (6)

- As with the mass equation, practical analysis is often facilitated as averages across inlets and exits

$$Q_{net,in} + W_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\cancel{V} + \sum_{out} \dot{m}\left(\frac{P}{\rho} + e\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + e\right)$$

$$m = \int_{A_C} \rho\left(\vec{V}\cdot\vec{n}\right) dA_c$$

- Since $e=u+ke+pe = u+V^2/2+gz$

$$Q_{net,in} + W_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\cancel{V} + \sum_{out} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right)$$

2-4 The First Law of Thermodynamics (7)

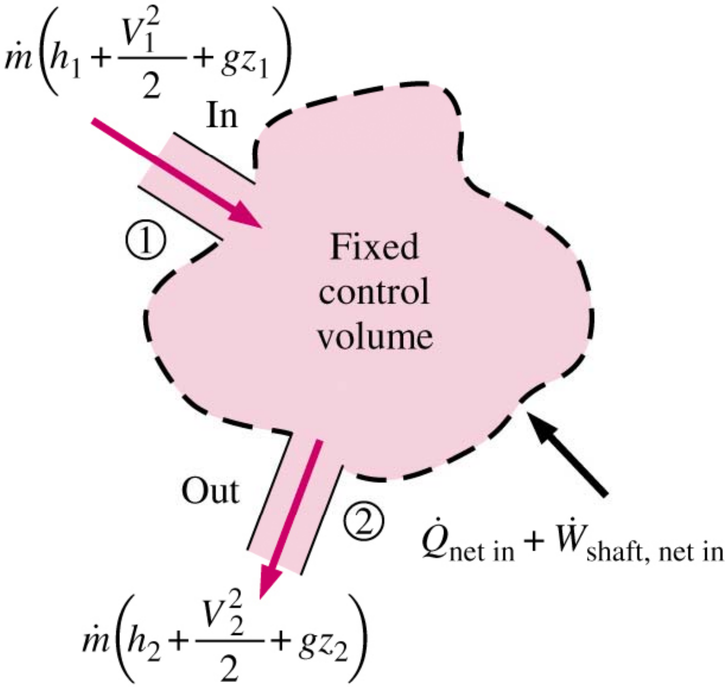
Energy Analysis of Steady Flows:

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

- For steady flow, time rate of change of the energy content of the CV is zero.
- This equation states: *the net rate of energy transfer to a CV by heat and work transfers during steady flow is equal to the difference between the rates of outgoing* and incoming energy flows with mass.

2-4 The First Law of Thermodynamics (8)

Energy Analysis of Steady Flows:



- For **single-stream devices**, mass flow rate is constant.

$$q_{net,in}+w_{shaft,net,in}=h_2-h_1+\frac{V_2^2-V_1^2}{2}+g\left(z_2-z_1\right)$$

$$w_{shaft,net,in}+\frac{P_1}{\rho_1}+\frac{V_1^2}{2}+gz_1=\frac{P_2}{\rho_2}+\frac{V_2^2}{2}+gz_2+\left(u_2-u_1-q_{net,in}\right)$$

$$\frac{P_1}{\rho_1}+\frac{V_1^2}{2}+gz_1+w_{pump}=\frac{P_2}{\rho_2}+\frac{V_2^2}{2}+gz_2+w_{turbine}+e_{mech,loss}$$

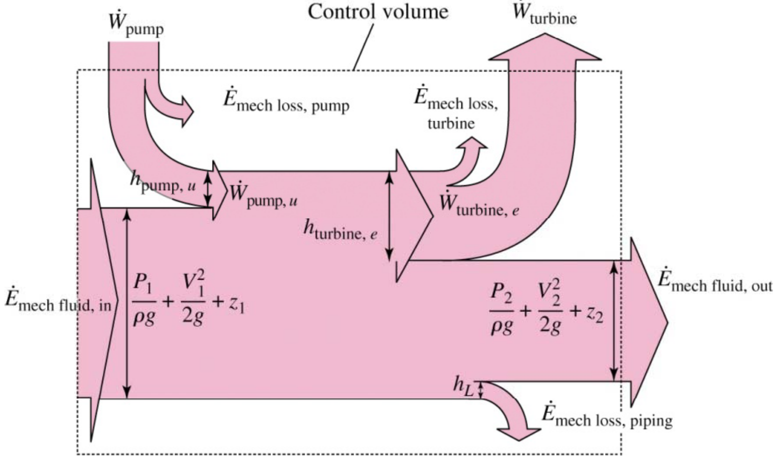
2-4 The First Law of Thermodynamics (9)

Energy Analysis of Steady Flows:

- Divide by *g* to get each term in units of length

$$\frac{P_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1+h_{pump}=\frac{P_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2+h_{turbine}+h_L$$

Magnitude of each term is now expressed as an equivalent column height of fluid, i.e., *Head*



2-4 The First Law of Thermodynamics (10)

- If we neglect piping losses, and have a system without pumps or turbines

$$\frac{P_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1=\frac{P_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2$$

- This is the **Bernoulli equation**
- 3 terms correspond to: Static, dynamic, and hydrostatic head (or pressure).

- Limitations on the use of the Bernoulli Equation
  - Steady flow: $d/dt = 0$
  - Frictionless flow
  - No shaft work: $w_{pump}=w_{turbine}=0$
  - Incompressible flow: ρ = constant
  - No heat transfer: $q_{net,in}=0$
  - Applied along a streamline (except for irrotational flow)

2-5 The Second Law of Thermodynamics

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

$$S_{\text{system}} = \int_{M(\text{system})} s\,dm = \int_{\forall(\text{system})} s\,\rho\,d\forall$$

$$\left.\frac{dS}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} s\,\rho\,d\forall + \int_{\text{CS}} s\,\rho\vec{V}\cdot d\vec{A}$$

$$\frac{\partial}{\partial t}\int_{\text{CV}} s\,\rho\,d\forall + \int_{\text{CS}} s\,\rho\,\vec{V}\cdot d\vec{A} \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right)dA$$