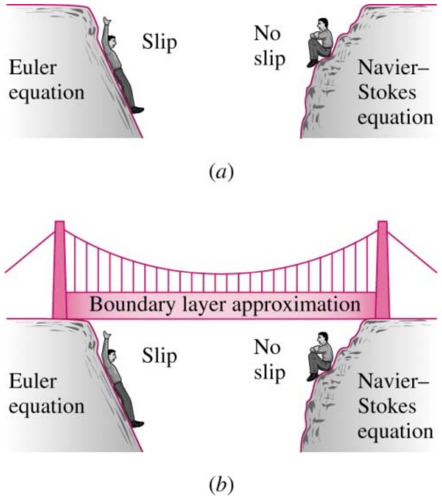
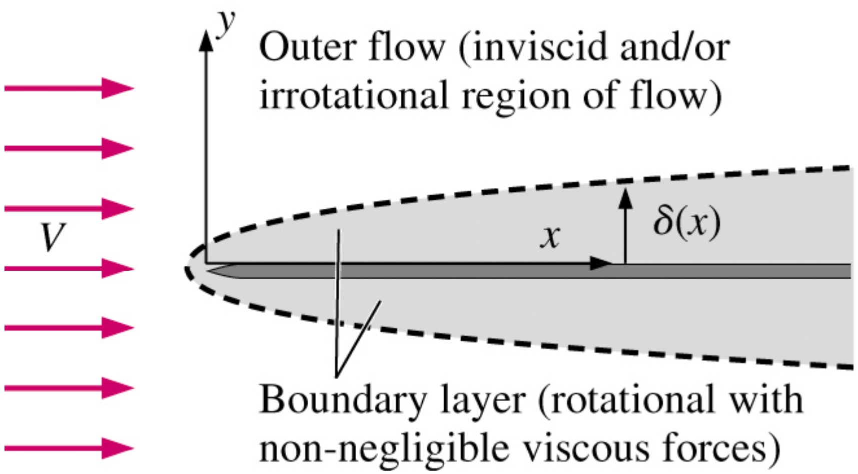
Chapter 4: Boundary Layer

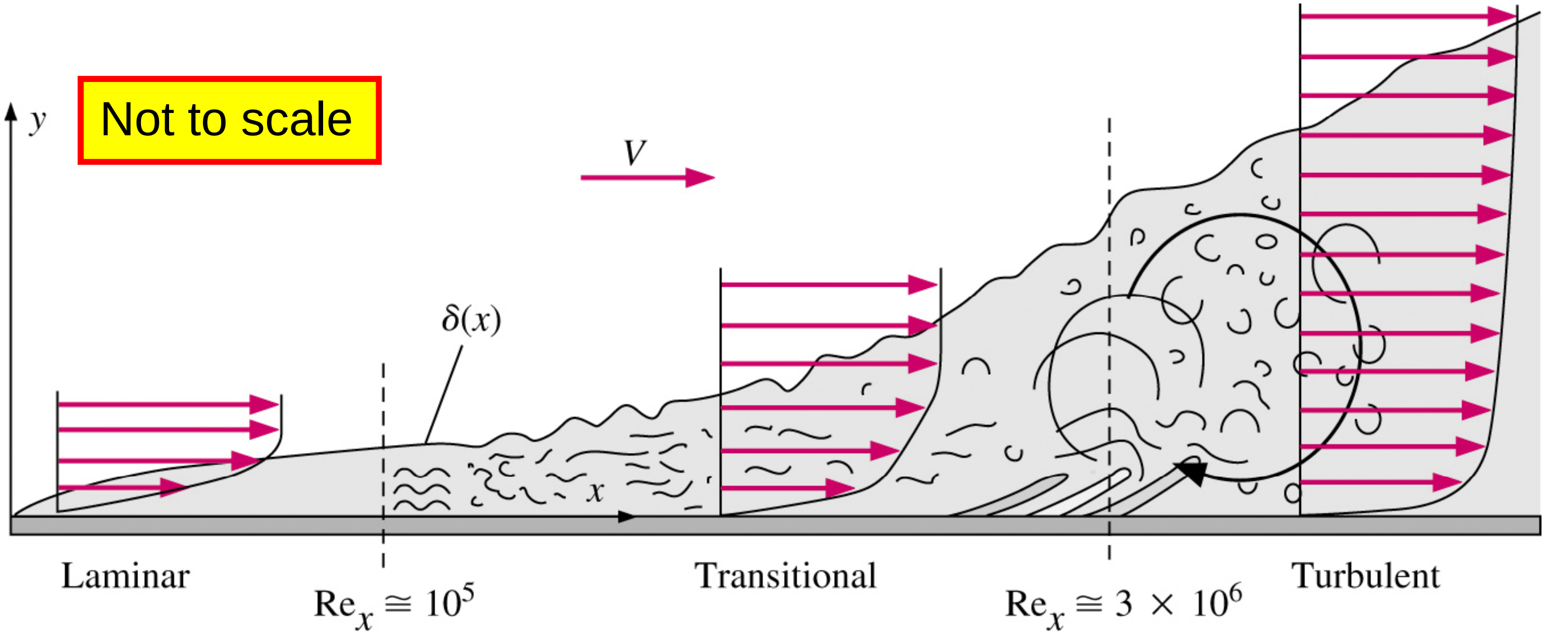
Boundary Layer (BL) Approximation

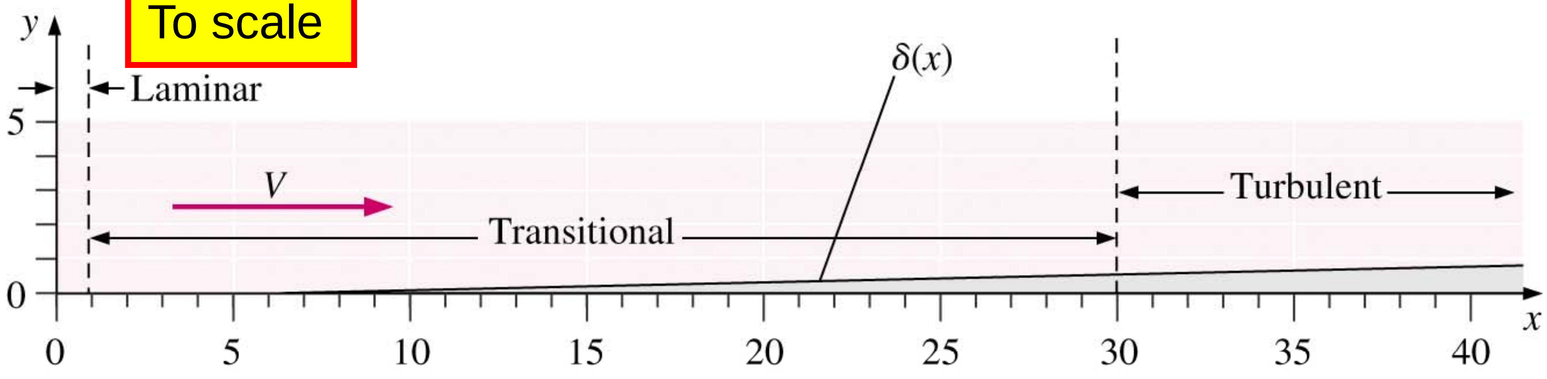




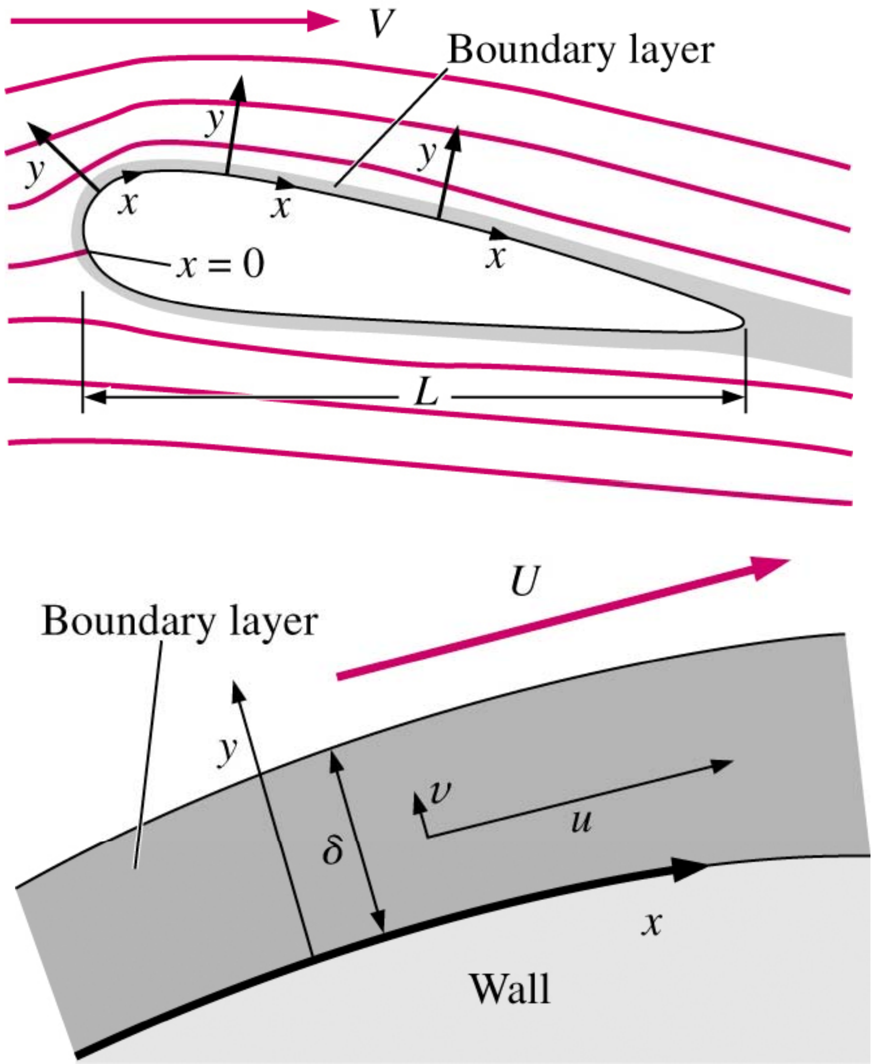
- BL approximation bridges the gap between the Euler and NS equations, and between the slip and no-slip BC at the wall.
- Prandtl (1904) introduced the BL approximation

Boundary Layer (BL) Approximation





Boundary Layer (BL) Approximation



- BL Equations: we restrict attention to steady, 2D, laminar flow (although method is fully applicable to unsteady, 3D, turbulent flow)
- BL coordinate system
  - x : tangential direction
  - y : normal direction

Boundary Layer (BL) Approximation

- To derive the equations, start with the steady nondimensional NS equations

$$\left(\vec{V}^{\star} \cdot \nabla^{\star}\right) \vec{V}^{\star} = -\left[Eu\right] \nabla^{\star} P^{\star} + \left[\frac{1}{Re}\right] \nabla^{\star 2} \vec{V}^{\star}$$

- Recall definitions $Eu = \dfrac{P_0 - P_\infty}{\rho V^2}, \quad Re = \dfrac{\rho V L}{\mu}$
- Since $\rho V^2 \sim P - P_\infty$, Eu ~ 1
- Re >> 1, Should we neglect viscous terms? No!, because we would end up with the Euler equation along with deficiencies already discussed.
- Can we neglect **some** of the viscous terms?

Boundary Layer (BL) Approximation

- To answer question, we need to redo the nondimensionalization
  - Use L as length scale in streamwise direction and for derivatives of velocity and pressure with respect to x.
  - Use δ (boundary layer thickness) for distances and derivatives in y.
  - Use local outer (or edge) velocity U_e.

Boundary Layer (BL) Approximation

- Orders of Magnitude (OM)

$$U \sim 1, \quad P - P_\infty \sim \rho U^2, \quad \frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

- What about *V*? Use continuity

$$\left.\underbrace{\frac{\partial U}{\partial x}}_{\sim U/L} + \underbrace{\frac{\partial V}{\partial y}}_{\sim V/\delta} = 0 \right\} V \sim \frac{U\delta}{L}$$

- Since $\delta / L \ll 1 \rightarrow V \ll U$

Boundary Layer (BL) Approximation

- Now, define new nondimensional variables

$$x^\star = \frac{x}{L}, \quad y^\star = \frac{y}{\delta}, \quad U^\star = \frac{U}{U_e}, \quad V^\star = \frac{VL}{U_e\delta}, \quad P^\star = \frac{P - P_\infty}{\rho U_e^2}$$

- All are order unity, therefore normalized
- Apply to *x*- and *y*-components of NSE
- Go through details of derivation on blackboard.

Boundary Layer (BL) Approximation

- Incompressible Laminar Boundary Layer Equations

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

X-Momentum

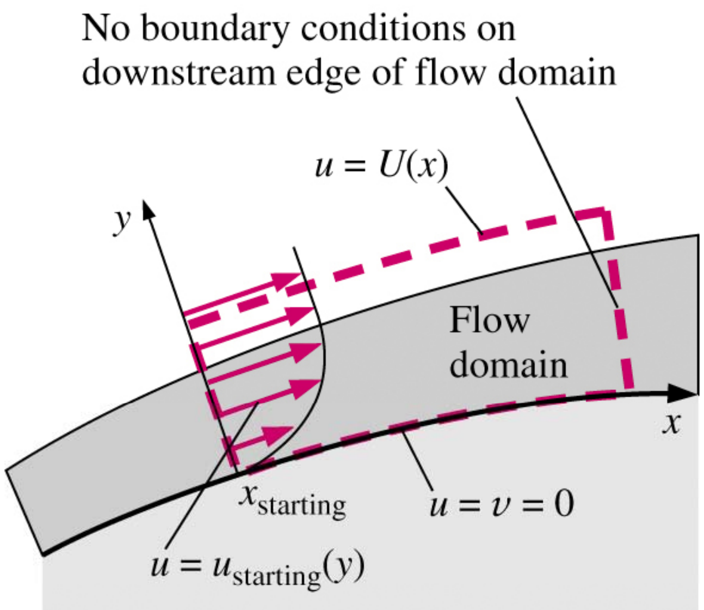
$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = U_e\frac{\partial U_e}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2}$$

Y-Momentum

$$\frac{\partial P}{\partial y} = 0$$

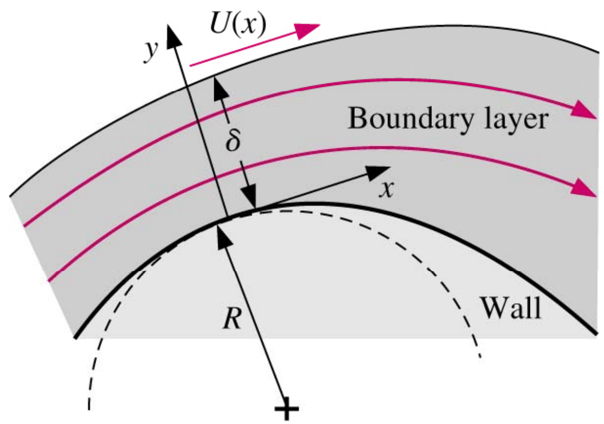
Boundary Layer Procedure

1. Solve for outer flow, ignoring the BL. Use potential flow (irrotational approximation) or Euler equation
2. Assume $\delta/L \ll 1$ (thin BL)
3. Solve BLE
   - $y = 0 \Rightarrow$ no-slip, u=0, v=0
   - $y = \delta \Rightarrow U = U_e(x)$
   - $x = x_0 \Rightarrow u = u(x_0)$, $v=v(x_0)$
4. Calculate δ, θ, δ^*, τ_w, Drag
5. Verify $\delta/L \ll 1$
6. If δ/L is not $\ll 1$, use δ^* as body and goto step 1 and repeat



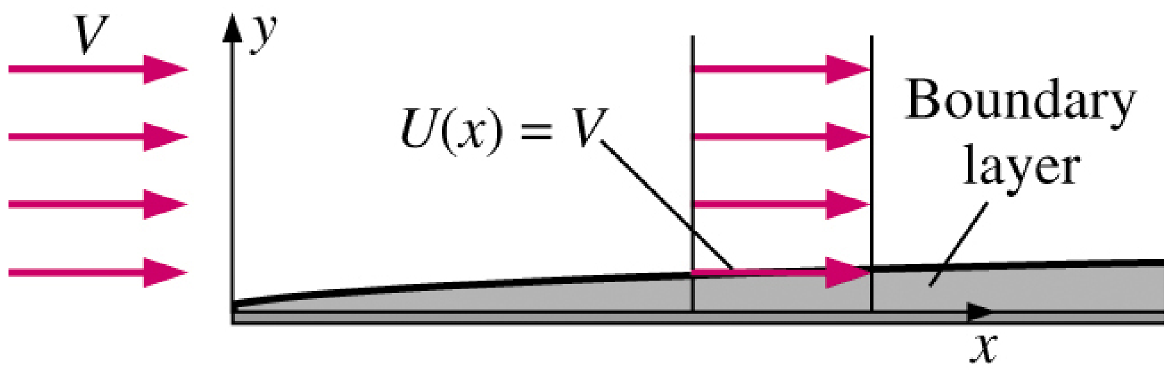
Boundary Layer Procedure

- Possible Limitations
  1. Re is not large enough $\Rightarrow$ BL may be too thick for thin BL assumption.
  2. *$\partial p/\partial y \neq 0$ due to wall curvature $\delta \sim R$*
  3. Re too large $\Rightarrow$ turbulent flow at Re = 1x10^5. BL approximation still valid, but new terms required.
  4. Flow separation

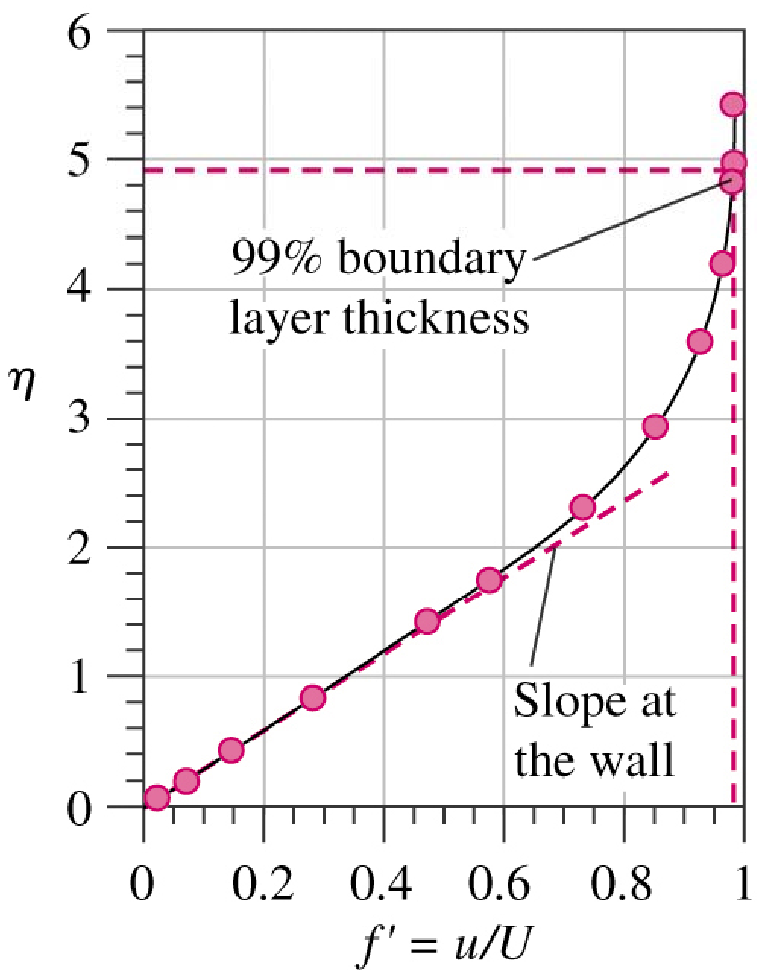


Boundary Layer Procedure

- Before defining and δ^* and θ, are there analytical solutions to the BL equations?
  - *Unfortunately, NO*
- *Blasius Similarity Solution* boundary layer on a flat plate, constant edge velocity, zero external pressure gradient



Blasius Similarity Solution



- Blasius introduced similarity variables

$$f' = \frac{U}{U_e} \qquad \eta = y\sqrt{\frac{U_e}{\nu x}}$$

- This reduces the BLE to

$$f''' + f f'' = 0$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

- This ODE can be solved using Runge-Kutta technique
- Result is a BL profile which holds at every station along the flat plate

Blasius Similarity Solution

TABLE 10–3

Solution of the Blasius laminar flat plate boundary layer in similarity variables*

η	f''	f'	f	η	f''	f'	f
0.0	0.33206	0.00000	0.00000	2.4	0.22809	0.72898	0.92229
0.1	0.33205	0.03321	0.00166	2.6	0.20645	0.77245	1.07250
0.2	0.33198	0.06641	0.00664	2.8	0.18401	0.81151	1.23098
0.3	0.33181	0.09960	0.01494	3.0	0.16136	0.84604	1.39681
0.4	0.33147	0.13276	0.02656	3.5	0.10777	0.91304	1.83770
0.5	0.33091	0.16589	0.04149	4.0	0.06423	0.95552	2.30574
0.6	0.33008	0.19894	0.05973	4.5	0.03398	0.97951	2.79013
0.8	0.32739	0.26471	0.10611	5.0	0.01591	0.99154	3.28327
1.0	0.32301	0.32978	0.16557	5.5	0.00658	0.99688	3.78057
1.2	0.31659	0.39378	0.23795	6.0	0.00240	0.99897	4.27962
1.4	0.30787	0.45626	0.32298	6.5	0.00077	0.99970	4.77932
1.6	0.29666	0.51676	0.42032	7.0	0.00022	0.99992	5.27923
1.8	0.28293	0.57476	0.52952	8.0	0.00001	1.00000	6.27921
2.0	0.26675	0.62977	0.65002	9.0	0.00000	1.00000	7.27921
2.2	0.24835	0.68131	0.78119	10.0	0.00000	1.00000	8.27921

* η is the similarity variable defined in Eq. 4 above, and function $f(\eta)$ is solved using the Runge–Kutta numerical technique. Note that f'' is proportional to the shear stress τ, f' is proportional to the x-component of velocity in the boundary layer ($f' = u/U$), and f itself is proportional to the stream function. f' is plotted as a function of η in Fig. 10–99.

Blasius Similarity Solution

- Boundary layer thickness can be computed by assuming that δ corresponds to point where U/U_e = 0.990. At this point, η = 4.91, therefore

$$\eta = 4.91 = \sqrt{\frac{U_e}{\nu x}}\delta \Longrightarrow \frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$$

Recall

$$Re_x = \frac{\rho U x}{\mu}$$

- Wall shear stress τ_w and friction coefficient $C_{f,x}$ can be directly related to Blasius solution

$$\tau_w = \mu \left.\frac{\partial U}{\partial y}\right|_{y=0} = f''(0)\frac{\rho U_e^2}{\sqrt{Re_x}} = 0.332\frac{\rho U_e^2}{\sqrt{Re_x}}$$

$$C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re_x}}$$

Displacement Thickness

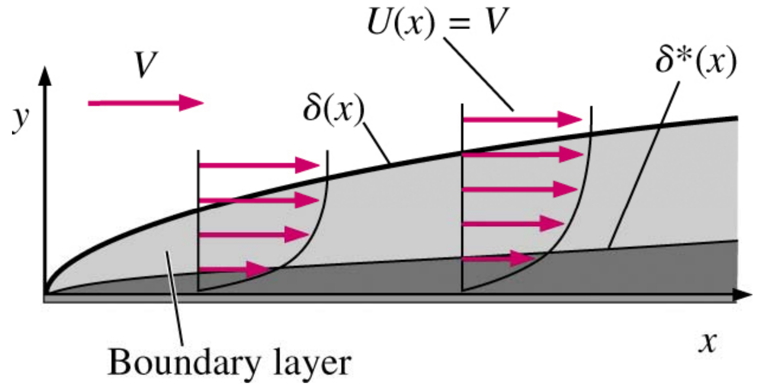
- Displacement thickness δ^* is the imaginary increase in thickness of the wall (or body), as seen by the outer flow, and is due to the effect of a growing BL.
- Expression for δ^* is based upon control volume analysis of conservation of mass

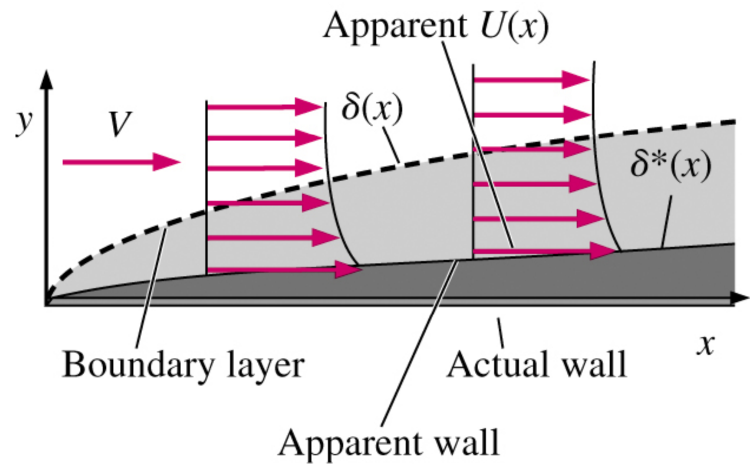
$$\delta^\star = \int_0^\infty \left(1 - \frac{U}{U_e}\right) \mathrm{d}y$$

- Blasius profile for laminar BL can be integrated to give

$$\frac{\delta^\star}{x} = \frac{1.72}{\sqrt{Re_x}}$$

(≈1/3 of δ)





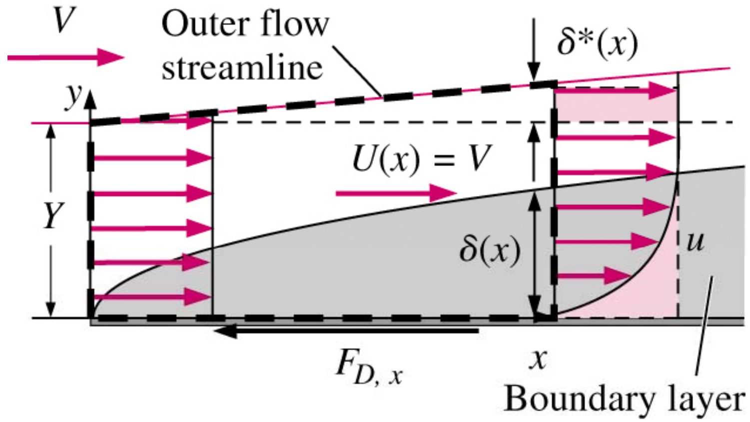
Momentum Thickness

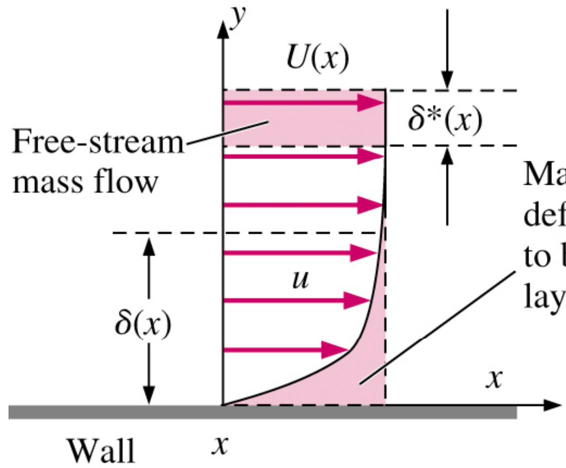
- Momentum thickness θ is another measure of boundary layer thickness.
- Defined as the loss of momentum flux per unit width divided by ρU^2 due to the presence of the growing BL.
- Derived using CV analysis.

$$\theta = \int_0^\infty \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) \mathrm{d}y = \frac{F_{D,x}}{\rho U_e^2 w}$$

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

θ for Blasius solution, identical to $C_{f,x}$





Turbulent Boundary Layer

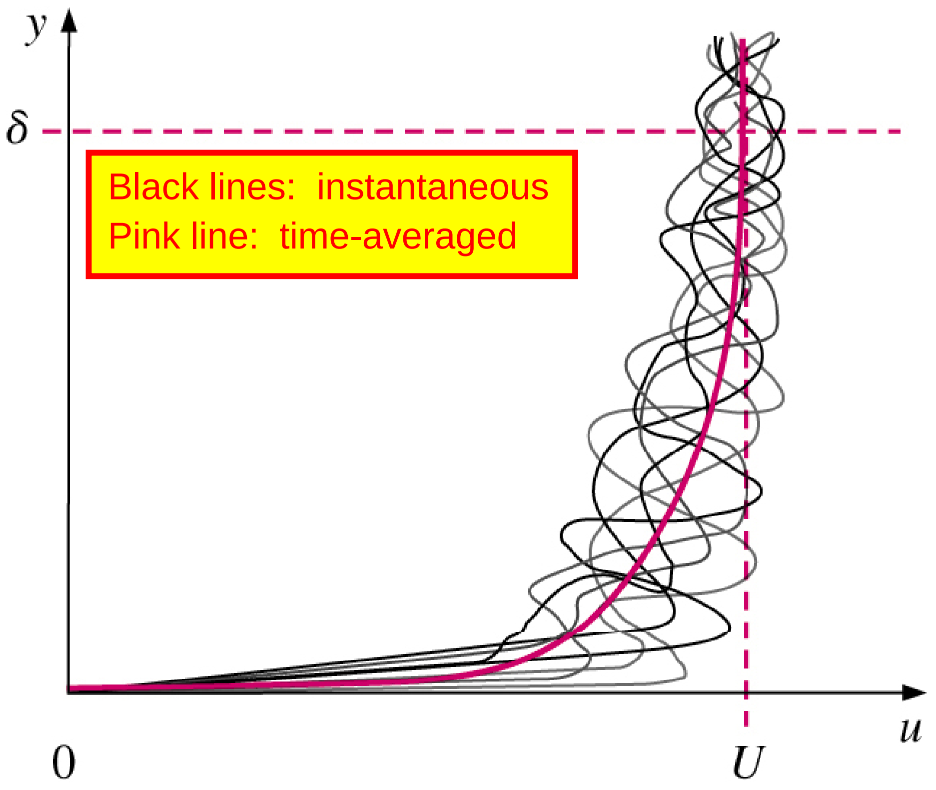
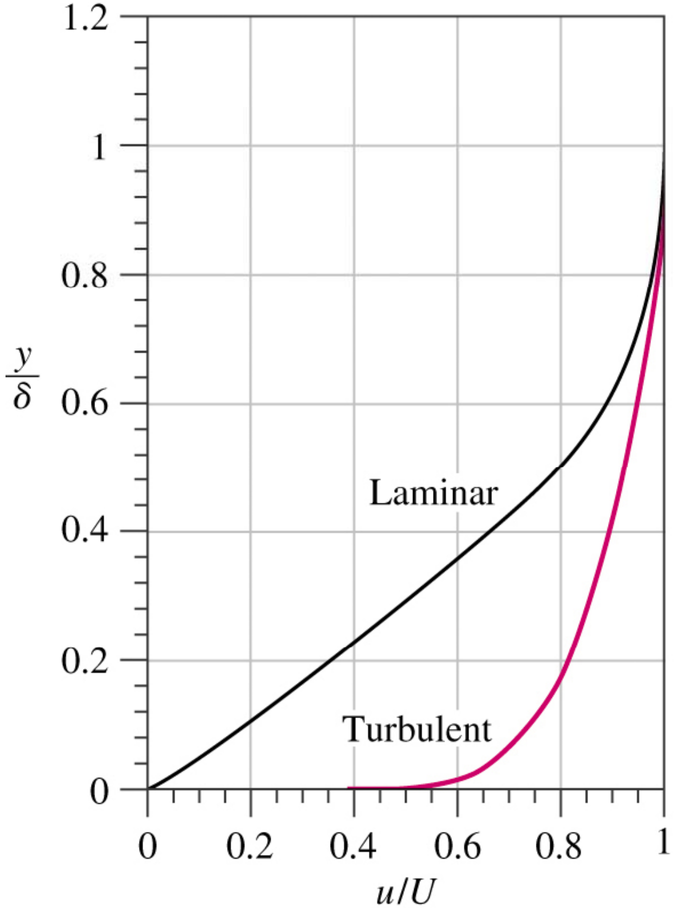


Illustration of unsteadiness of a turbulent BL



Comparison of laminar and turbulent BL profiles

Turbulent Boundary Layer

- All BL variables [$U(y)$, δ, δ^*, θ] are determined empirically.
- One common empirical approximation for the time-averaged velocity profile is the **one-seventh-power law**

$$\frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/7} \qquad y \leq \delta$$

$$\frac{U}{U_e} \cong 1 \qquad y > \delta$$

Turbulent Boundary Layer

TABLE 10–4

Summary of expressions for laminar and turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream*

Property	Laminar	(a) Turbulent(†)	(b) Turbulent(‡)
Boundary layer thickness	$\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$	$C_{f,x} \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$

* Laminar values are exact and are listed to three significant digits, but turbulent values are listed to only two significant digits due to the large uncertainty affiliated with all turbulent flow fields.

† Obtained from one-seventh-power law.

‡ Obtained from one-seventh-power law combined with empirical data for turbulent flow through smooth pipes.

Turbulent Boundary Layer

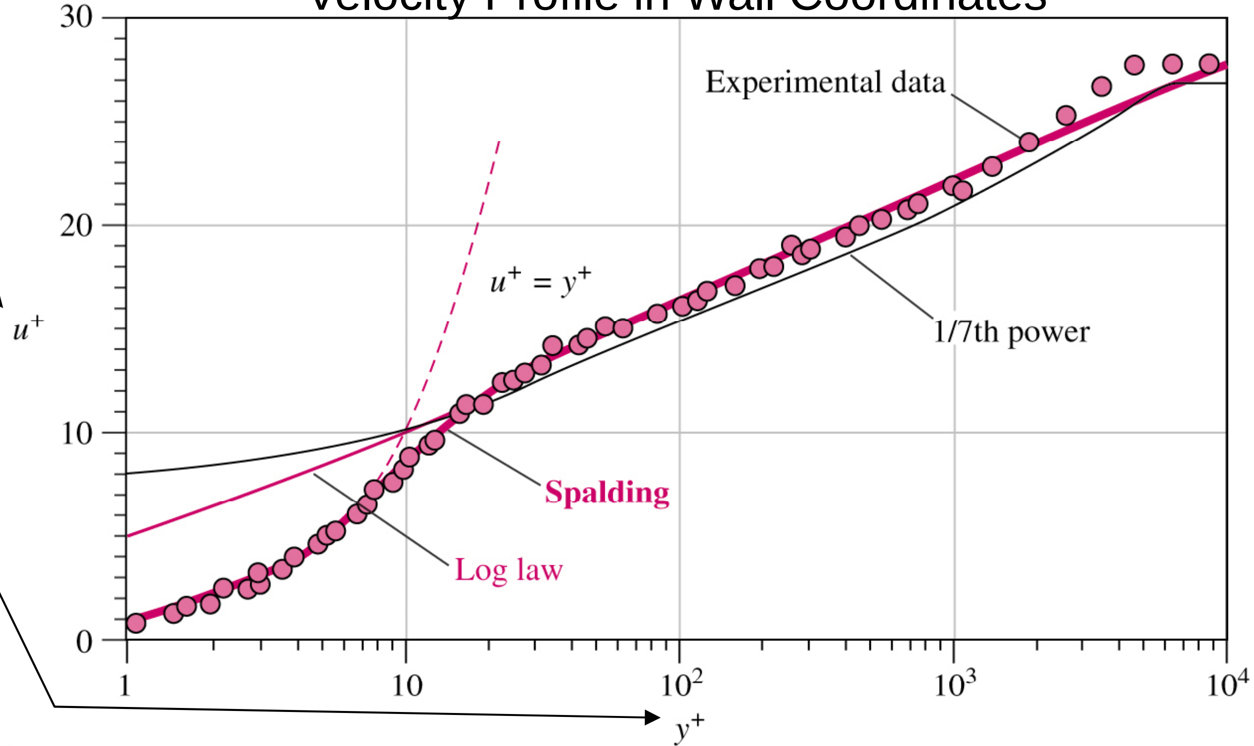
- Flat plate zero-pressure-gradient TBL can be plotted in a universal form if a new velocity scale, called the friction velocity U_τ, is used. Sometimes referred to as the "Law of the Wall"

$$u^+ = \frac{U}{U_\tau}$$

$$y^+ = \frac{U_\tau y}{\nu}$$

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

Velocity Profile in Wall Coordinates



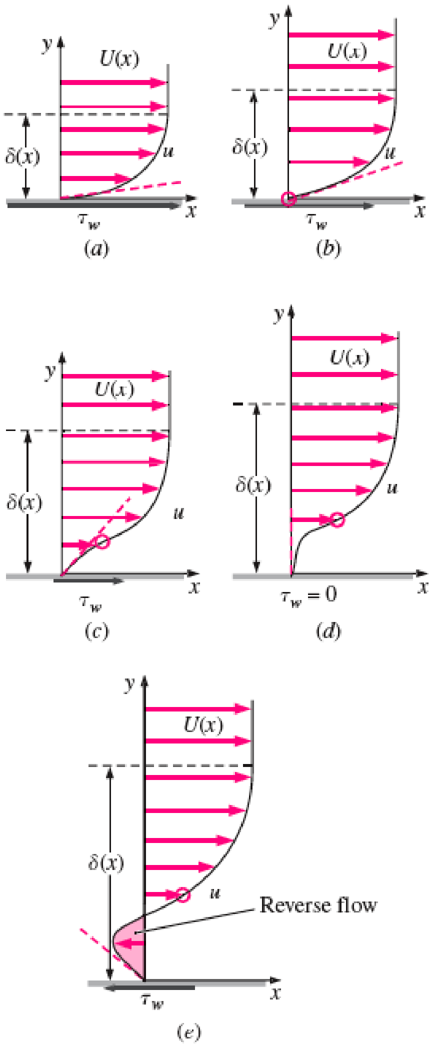
Turbulent Boundary Layer

- Despite it's simplicity, the Law of the Wall is the basis for many CFD turbulence models.
- Spalding (1961) developed a formula which is valid over most of the boundary layer

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right]$$

  - κ, B are constants

Pressure Gradients



- Shape of the BL is strongly influenced by external pressure gradient
  - (a) favorable ($dP/dx < 0$)
  - (b) zero
  - (c) mild adverse ($dP/dx > 0$)
  - (d) critical adverse ($\tau_w = 0$)
  - (e) large adverse with reverse (or separated) flow

Pressure Gradients

- The BL approximation is not valid downstream of a separation point because of reverse flow in the separation bubble.
- Turbulent BL is more resistant to flow separation than laminar BL exposed to the same adverse pressure gradient