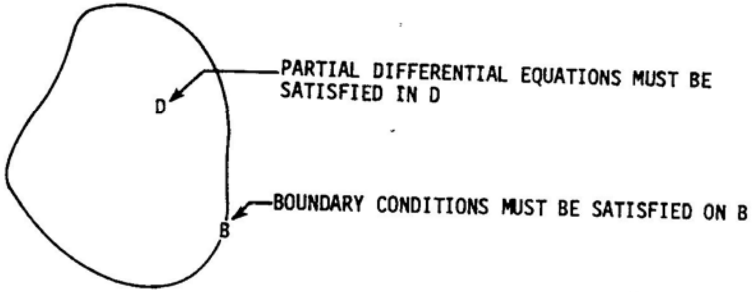
Chapter 5 Types of Governing Equations

Types of Governing Equations (1)

Physical Classification-1

- Equilibrium problems:



(1) They are problems in which a solution of a given PDE is desired in a closed domain subject to a prescribed set of boundary conditions. **Equilibrium problems are boundary value problems.**

(2) Sometimes, equilibrium problems are referred to as **jury problems**,since the **solution of the PDE** at every point in the domain **depends upon the prescribed boundary condition** at every point on B.

(3) Mathematically, **equilibrium problems are governed by elliptic PDEs.**

Types of Governing Equations (2)

Physical Classification-2

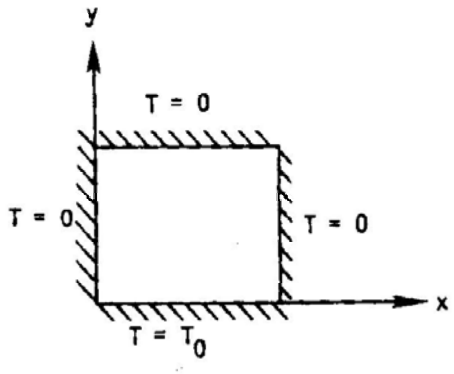
(4) Example 1 of equilibrium problem:

The steady-state temperature distribution in a conducting medium is governed by Laplace's equation.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad 0 \le x \le 1; 0 \le y \le 1$$

with boundary conditions

$T(0,y)=0$, $T(1,y)=0$, $T(x,0)=T_0$, $T(x,1)=0$

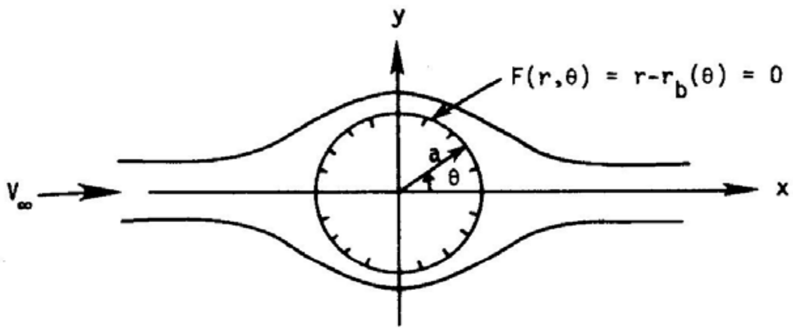


Types of Governing Equations (3)
Physical Classification-3

(5) Example 2 of equilibrium problem:

The irrotational flow of an incompressible inviscid fluid is governed by Laplace's eq.

$$\nabla^2 \phi = 0$$

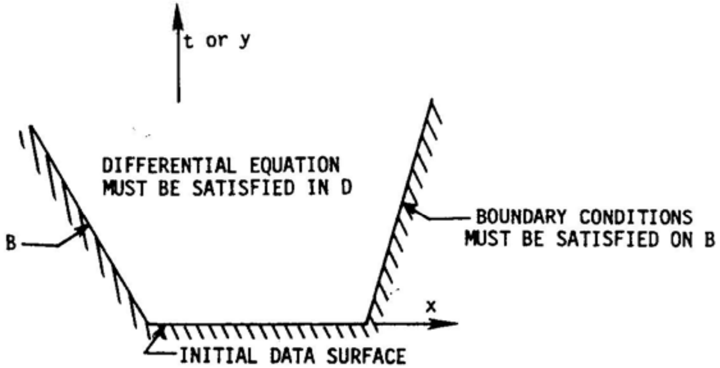


Types of Governing Equations (4)

Physical Classification-4

- Marching Problems:

(1) **Marching or propagation problems are transient or transient-like problems** where the solution of a PDE is required on an open domain subject to **a set of initial conditions** and **a set of boundary conditions**.



(2) Problems in this category are **initial value or initial-boundary** problems. The solution must be computed by marching outward from the initial data surface while satisfying the boundary conditions.

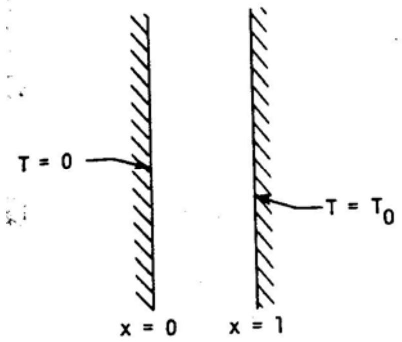
(3) Mathematically, these problems are governed by either **hyperbolic** or **parabolic** PDEs.

Types of Governing Equations (5)

Physical Classification-5

(4) Example of marching problem:

Determine the transient temperature distribution in a 1-D solid, as shown in the following figure.



(5) Typical examples of marching problems include unsteady inviscid flow, steady supersonic inviscid flow, transient heat conduction and boundary-layer flow.

Types of Governing Equations (6)

Mathematical Classification-1

- The general second-order PDE can be expressed as

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g(x,y)$$

  where a,b,c,d,e, and f are functions of (x,y), i.e., we consider a linear equation.

- The classification of a second-order PDE depends only on the second-derivative terms of the equation, so we may rearrange the above equation as

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = -(d\phi_x + e\phi_y + f\phi - g) = H$$

- As in the classification of general second-degree equations in analytic geometry, the PDE is classified as

  (1) Hyperbolic if $b^2-4ac > 0$

  (2) Parabolic if $b^2-4ac = 0$

  (3) Elliptic if $b^2-4ac < 0$

Types of Governing Equations (7)

Mathematical Classification-2

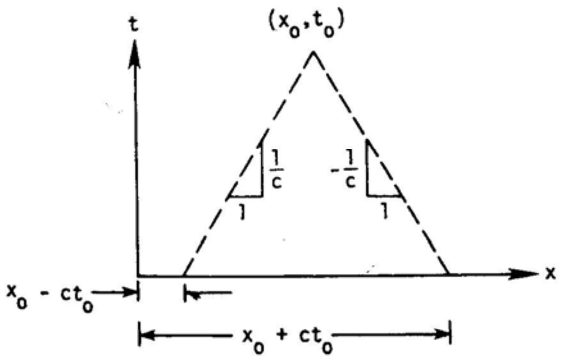
- Example of Hyperbolic PDE:

on the interval $-\infty < x < \infty$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ with initial condition $u(x,0)=f(x)$, $u_t(x,0)=g(x)$

Solution:

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$



Types of Governing Equations (8)
Mathematical Classification-3

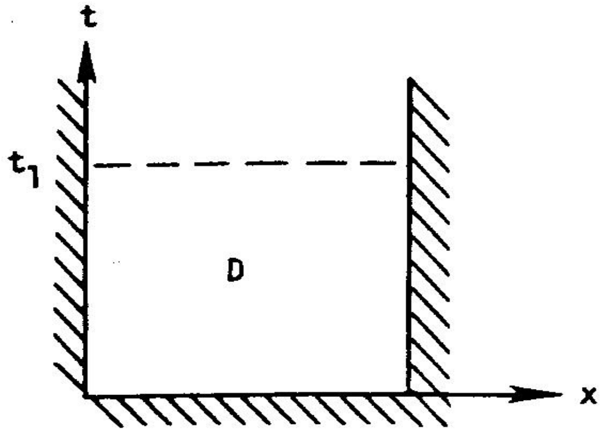
- Example of Parabolic PDE:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$u(0, y) = 0$$

$$u(t,0) = U \qquad t > 0$$

$$u(t,\infty) = 0$$



Types of Governing Equations (9)

Mathematical Classification-4

- **<u>Parabolic PDEs are associated with diffusion processes</u>**. The solutions of parabolic equations clearly show this behavior. **<u>While the PDEs controlling diffusion are marching problems</u>**, i.e., we solve them starting at some initial data plane and march forward in time or in a time-like direction, they do not exhibit the limited zones of influence that hyperbolic equations have. In contrast, the solution of a parabolic equation at time t_1 depends upon the entire physical domain ($t \leq t_1$), including any side boundary conditions.

Types of Governing Equations (10)

Mathematical Classification-5

- Example of Elliptic PDEs:

Given Laplace's equation on the unit disk

$$\nabla^2 u = 0 \qquad 0 \le r < 1 \qquad -\pi \le \theta < \pi$$

subject to boundary conditions

$$\frac{\partial u}{\partial r}(1,\theta) = f(\theta) \qquad -\pi \le \theta < \pi$$

Types of Governing Equations (11)

Mathematical Classification-6

- **Domain of Dependence:**

Fig. 2.9 Domain of dependence and zone of influence for an hyperbolic problem

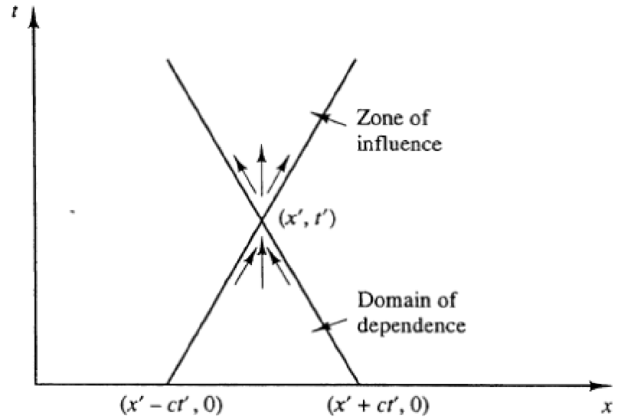
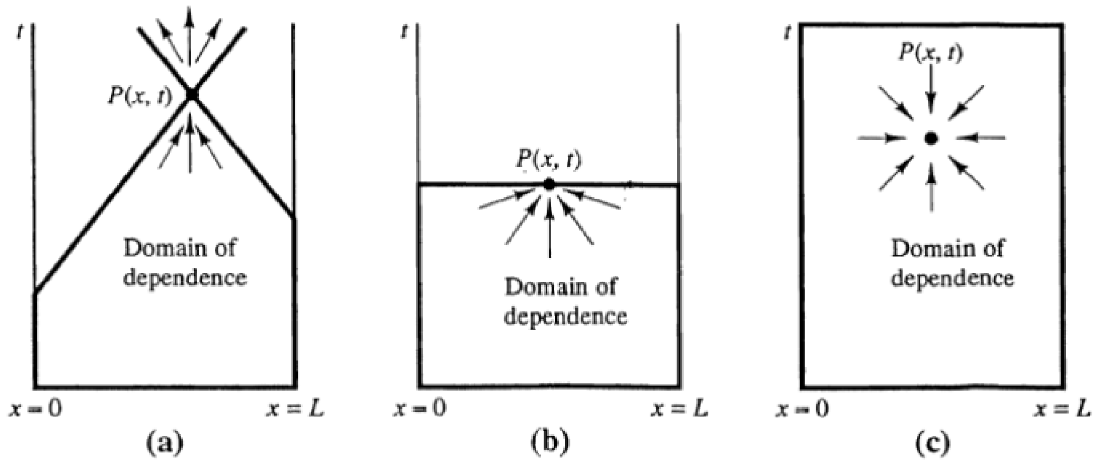


Fig. 2.10 Domains of dependence for
(a) hyperbolic,
(b) parabolic and
(c) elliptic problem



Types of Governing Equations (12)

Mathematical Classification-7

Table 2.2 Classification of physical behaviours

Problem type	*Equation type*	*Prototype equation*	*Conditions*	*Solution domain*	*Solution smoothness*
Equilibrium problems	Elliptic	$div\ grad\ \phi = 0$	Boundary conditions	Closed domain	Always smooth
Marching problems with dissipation	Parabolic	$\frac{\partial \phi}{\partial t} = \alpha\ div\ grad\ \phi$	Initial and boundary conditions	Open domain	Always smooth
Marching problems without dissipation	Hyperbolic	$\frac{\partial^2 \phi}{\partial t^2} = c^2\ div\ grad\ \phi$	Initial and boundary conditions	Open domain	May be discontinuous

Types of Governing Equations (12)

The Well-Posed Problem-1

- In order for a problem involving a PDE to be **well-posed, the solution to the problem must exist, must be unique**, and **must depend continuously upon the initial or boundary data**.
- Initial and Boundary Conditions

  (1) In order to obtain a unique solution of a PDE, a set of supplementary conditions must be provided to determine the arbitrary functions which result from the integration of PDE. The supplementary conditions are classified as **boundary** or **initial conditions**.

  (2) An initial condition is a requirement for which the dependent variable is specified at some initial state.

  Ex: $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$, for t=0, T=f(x, y)

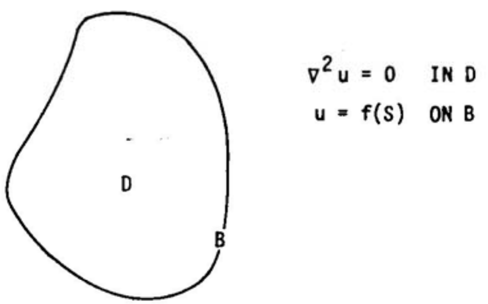
Types of Governing Equations (13)

The Well-Posed Problem-2

(3) A boundary condition is a requirement that the dependent variable or its derivative must satisfy on the boundary of the domain of the PDE.

(4) Various types of boundary conditions which will be encountered are

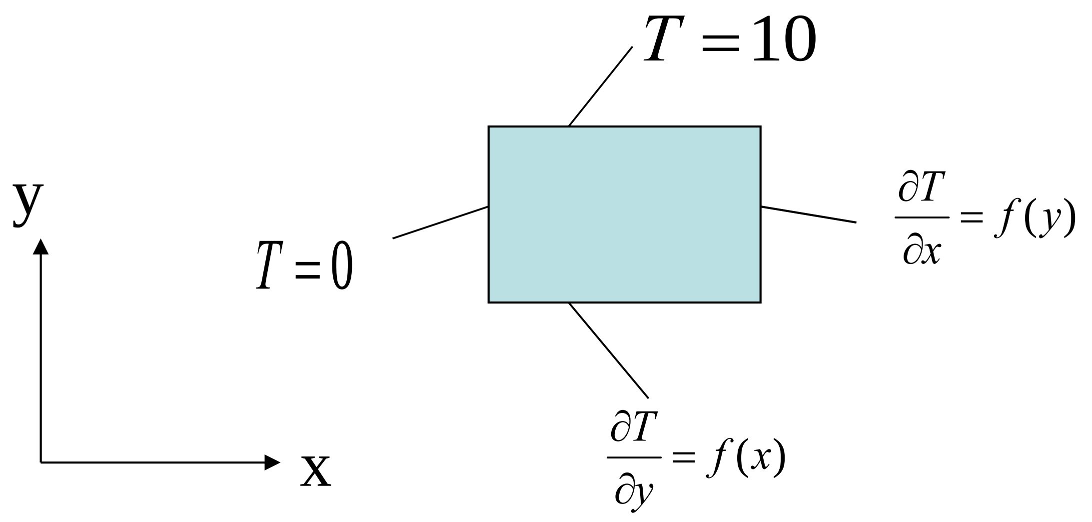
(a) The **Dirichlet** boundary condition: the dependent variable along the boundary is prescribed.



Types of Governing Equations (14)

The Well-Posed Problem-3

(b) The **Neumann** boundary condition: the normal gradient of the dependent variable along the boundary is specified.

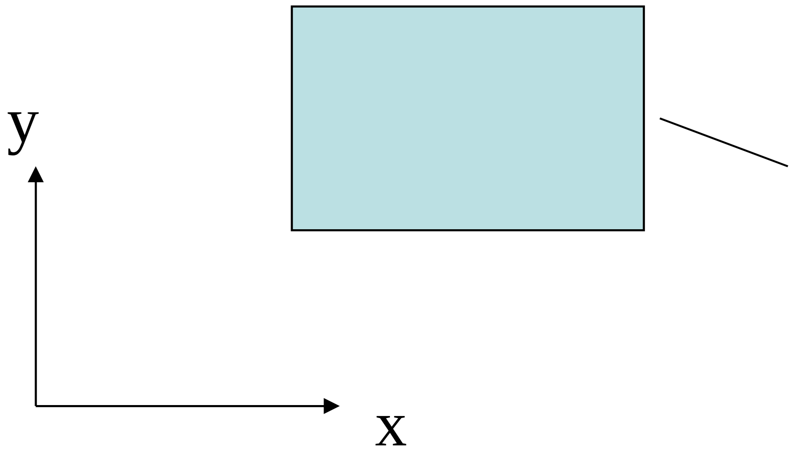


Types of Governing Equations (15)

The Well-Posed Problem-4

(c) The **Robin (or Mixed or Third)** boundary condition:

a combination of the function T and its normal derivative on the boundary.



$$-k\frac{\partial T}{\partial x} = h(T - T_{\infty})$$

Types of Governing Equations (16)

The Ill-Posed Problem-1

Example 1

$$\nabla^2 T = 0 \qquad for \qquad -\infty < x < \infty, \quad y \geq 0$$

- using separation of variables yields the solution

$$u = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

- However, when n is large,

$$u \rightarrow \frac{e^{ny}}{n^2} \rightarrow \infty, even \quad for\, small \quad y$$

- It violates the third requirement of well-posed problem.

Consequently, it is a ill-posed problem

Types of Governing Equations (17)

The Ill-Posed Problem-2

$$\nabla^2 T = 0$$

Subject to the Neumann condition for all the boundary

- ❑ The solution of this problem is multiple. Consequently, it is a ill-posed problem.