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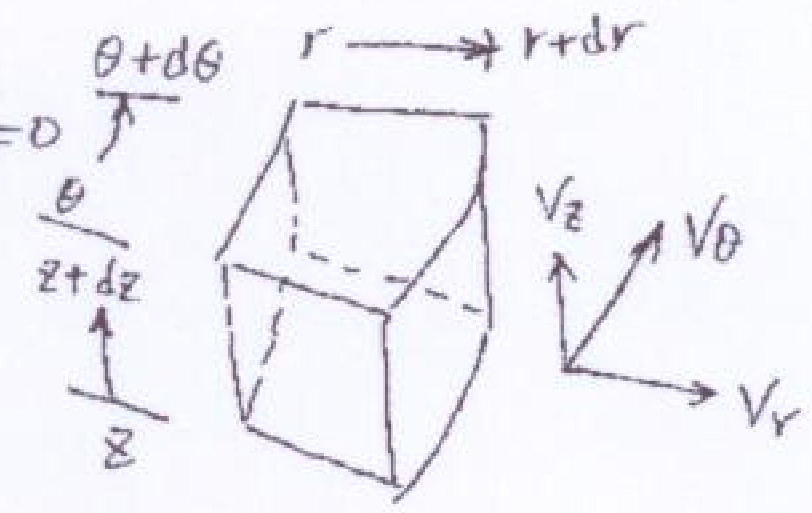
3. [Given]：如右圖所示

[Find]：

$$\frac{\partial}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)=0$$



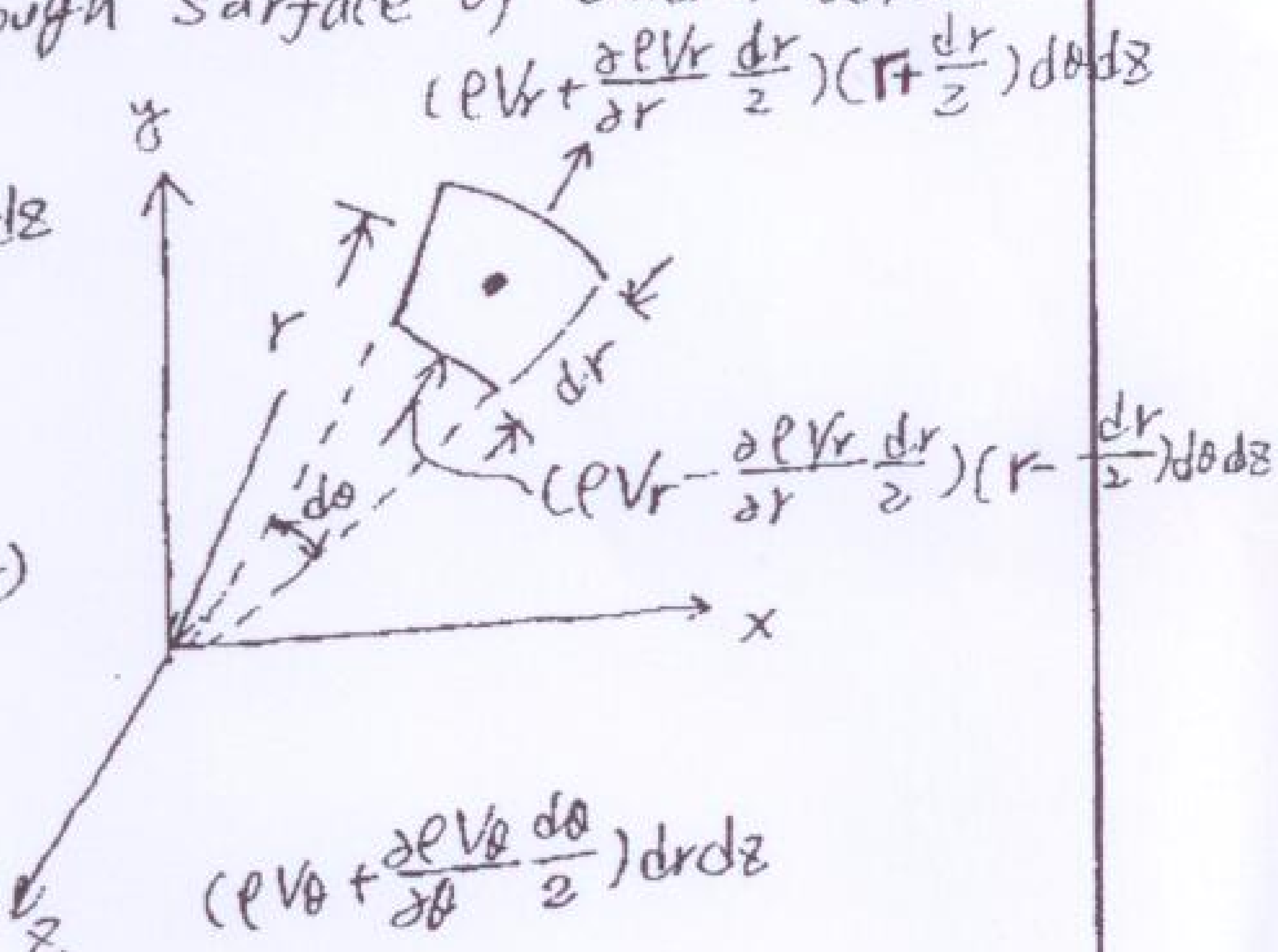
[Solution]：

連續方程式 (Continuity equation)

$$\frac{\partial}{\partial t}\int_{C\forall}\rho\, d\forall+\int_{CS}\rho\,\vec{v}\cdot\vec{n}\,dA=0$$

$$\frac{\partial}{\partial t}\int_{C\forall}\rho\, d\forall \approx \frac{\partial \rho}{\partial t}\, r\, d\theta\, dr\, dz \quad \text{——(1)}$$

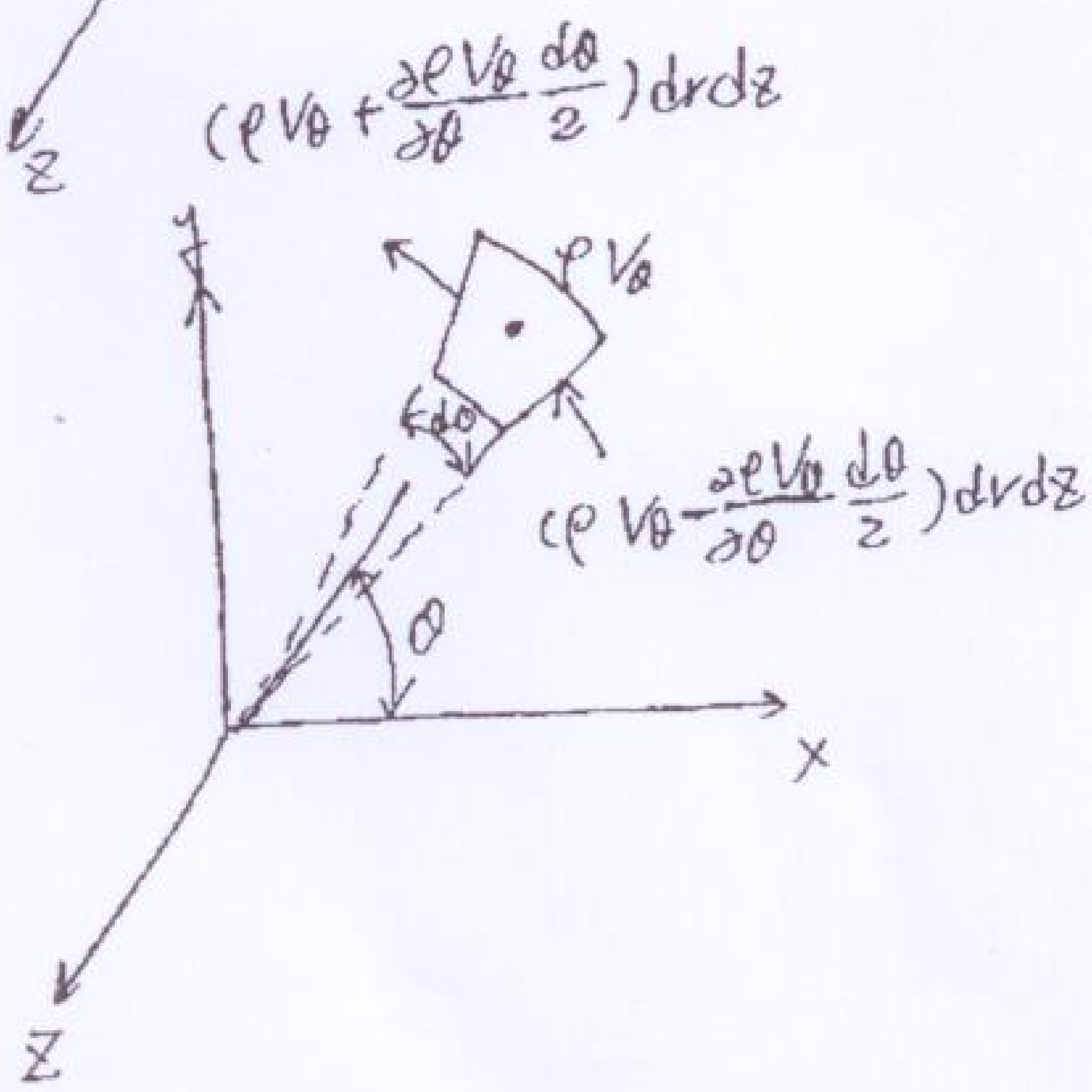
$\int_{CS}\rho\,\vec{v}\cdot\vec{n}\,dA$ = net rate of mass outflow through surface of control volume



$$r\text{-direction}=\left(\rho V_r+\frac{\partial \rho V_r}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta\, dz$$

$$-\left(\rho V_r-\frac{\partial \rho V_r}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta\, dz$$

$$=\frac{\partial \rho V_r}{\partial r}\, r\, dr\, d\theta\, dz+\rho V_r\, dr\, d\theta\, dz \quad \text{——(2)}$$



$$\theta\text{-direction}=\left(\rho V_\theta+\frac{\partial \rho V_\theta}{\partial \theta}\frac{d\theta}{2}\right)dr\, dz\ -$$

$$\left(\rho V_\theta-\frac{\partial \rho V_\theta}{\partial \theta}\frac{d\theta}{2}\right)dr\, dz$$

$$=\frac{\partial \rho V_\theta}{\partial \theta}\, dr\, d\theta\, dz \quad \text{——(3)}$$

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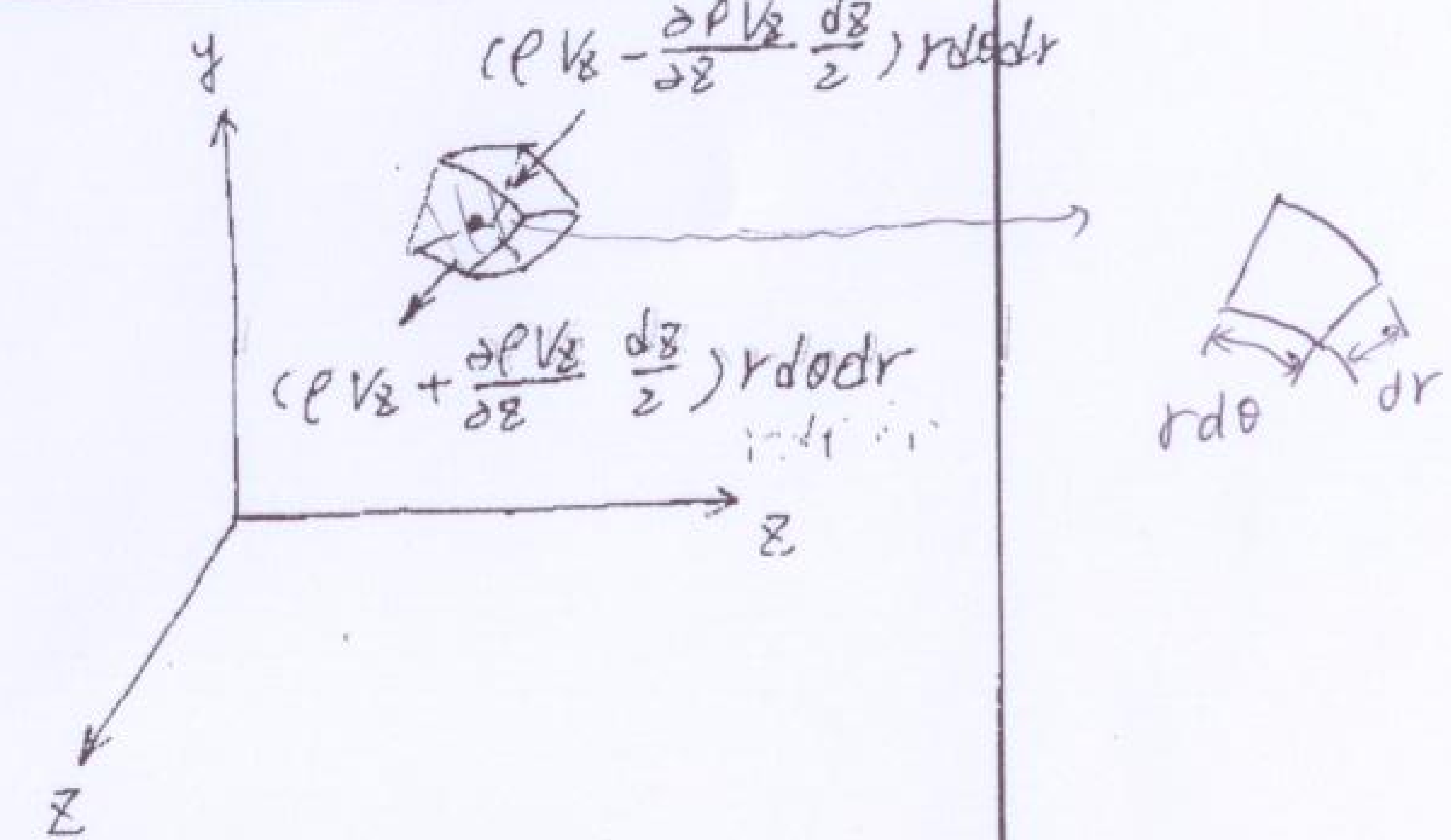
z - direction $= \left(\rho V_z + \frac{\partial \rho V_z}{\partial z}\frac{dz}{2}\right) r d\theta dr -$

$\left(\rho V_z - \frac{\partial \rho V_z}{\partial z}\frac{dz}{2}\right) r d\theta dr$

$= \frac{\partial \rho V_z}{\partial z} r\, dr d\theta dz$ ———————— (4)



$(1)+(2)+(3)+(4)=0$

$$\frac{\partial \rho}{\partial t} r d\theta dr dz + \frac{\partial \rho V_r}{\partial r} r dr d\theta dz + \rho V_r dr d\theta dz + \frac{\partial \rho V_\theta}{\partial \theta} dr d\theta dz + \frac{\partial \rho V_z}{\partial z} r dr d\theta dz = 0$$

divided by $r d\theta dr dz$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial \rho V_r}{\partial r} + \frac{\rho V_r}{r}}_{= \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial (\rho V_\theta)}{\partial \theta} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

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