Chapter 7: Dimensional Analysis and Modeling

7-1 Dimension and Units
7-2 Dimensional Homogeneity
7-3 Dimensional Analysis and Similarity
7-4 Methods of Repeating Variables
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7-1 Dimensions and Units (1)

- Review
  - Dimension: Measure of a physical quantity, e.g., length, time, mass
  - Units: Assignment of a number to a dimension, e.g., (m), (sec), (kg)
  - 7 Primary Dimensions:

1. Mass	m	(kg)
2. Length	L	(m)
3. Time	t	(sec)
4. Temperature	T	(K)
5. Current	I	(A)
6. Amount of Light	C	(cd)
7. Amount of matter	N	(mol)

7-1 Dimensions and Units (2)

- Review, continued
  - All non-primary dimensions can be formed by a combination of the 7 primary dimensions
  - Examples
    - {Velocity} = {Length/Time} = {L/t}
    - {Force} = {Mass Length/Time} = {mL/t^2}

7-2 Dimensional Homogeneity (1)

- Law of dimensional homogeneity (DH): every additive term in an equation must have the same dimensions
- Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

  - $\{p\} = \{\text{force/area}\} = \{\text{mass x length/time x 1/length}^2\} = \{m/(t^2 L)\}$
  - $\{1/2\rho V^2\} = \{\text{mass/length}^3 \text{ x (length/time)}^2\} = \{m/(t^2 L)\}$
  - $\{\rho g z\} = \{\text{mass/length}^3 \text{ x length/time}^2 \text{ x length}\} = \{m/(t^2 L)\}$

7-2 Dimensional Homogeneity (2)

Nondimensionalization of Equations

- Given the law of DH, if we divide each term in the equation by a collection of variables and constants that have the same dimensions, the equation is rendered nondimensional
- In the process of nondimensionalizing an equation, nondimensional parameters often appear, e.g., Reynolds number and Froude number

7-2 Dimensional Homogeneity (3)

- To nondimensionalize, for example, the Bernoulli equation, the first step is to list primary dimensions of all dimensional variables and constants

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

$\{p\} = \{m/(t^2L)\}$ $\{\rho\} = \{m/L^3\}$ $\{V\} = \{L/t\}$

$\{g\} = \{L/t^2\}$ $\{z\} = \{L\}$

- Next, we need to select Scaling Parameters. For this example, select L, U_0, ρ_0

7-2 Dimensional Homogeneity (4)

- By inspection, nondimensionalize all variables with scaling parameters

$$p^\star = \frac{p}{\rho_0 U_0^2} \qquad \rho^\star = \frac{\rho}{\rho_0} \qquad V^\star = \frac{V}{U_0}$$

$$g^\star = \frac{gL}{U_0^2} \qquad z^\star = \frac{z}{L}$$

- Back-substitute *p*, ρ, *V*, *g*, *z* into dimensional equation

$$\rho_0 U_0^2 p^\star + \frac{1}{2}\rho_0 \rho^\star \left(U_0^2 V^{\star 2}\right) + \rho_0 \rho^\star g^\star U_0^2 z^\star = C$$

7-2 Dimensional Homogeneity (5)

- Divide by $\rho_0 U_0^2$ and set $\rho^* = 1$ (incompressible flow)

$$p^\star + \frac{1}{2}V^{\star 2} + g^\star z^\star = \frac{C}{\rho_0 U_0^2} = C^\star$$

- Since $g^* = 1/Fr^2$, where $Fr = \frac{U_0}{\sqrt{gL}}$

$$p^\star + \frac{1}{2}V^{\star 2} + \frac{1}{Fr^2} z^\star = C^\star$$

7-2 Dimensional Homogeneity (6)

- Note that convention often dictates many of the nondimensional parameters, e.g., $1/2\rho_0 U_0^2$ is typically used to nondimensionalize pressure.

$$p^\star = \frac{p - p_0}{\frac{1}{2}\rho U_0^2}$$

- This results in a slightly different form of the nondimensional equation

$$p^\star + V^{\star 2} + \frac{2}{Fr^2} z^\star = C^\star$$

- BE CAREFUL! Always double check definitions.

7-2 Dimensional Homogeneity (7)

- Advantages of nondimensionalization
  - Increases insight about key parameters
  - Decreases number of parameters in the problem
    - Easier communication
    - Fewer experiments
    - Fewer simulations
  - Extrapolation of results to untested conditions

7-3 Dimensional Analysis and Similarity (1)

- Nondimensionalization of an equation is useful only when the equation is known!
- In many real-world flows, the equations are either unknown or too difficult to solve.
  - *Experimentation* is the only method of obtaining reliable information
  - In most experiments, geometrically-scaled models are used (time and money).
  - Experimental conditions and results must be properly scaled so that results are meaningful for the full-scale prototype.
  - ***Dimensional Analysis***

7-3 Dimensional Analysis and Similarity (2)

- Primary purposes of dimensional analysis
  - To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in reporting of results
  - To obtain scaling laws so that prototype performance can be predicted from model performance.
  - To predict trends in the relationship between parameters.

7-3 Dimensional Analysis and Similarity (3)

- **Geometric Similarity** - the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.
- **Kinematic Similarity** - velocity as any point in the model must be proportional
- **Dynamic Similarity** - *all forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow.
- **Complete Similarity** is achieved only if all 3 conditions are met. This is not always possible, e.g., river hydraulics models.

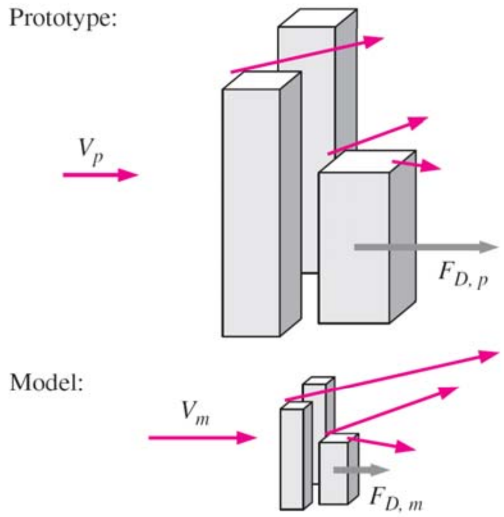
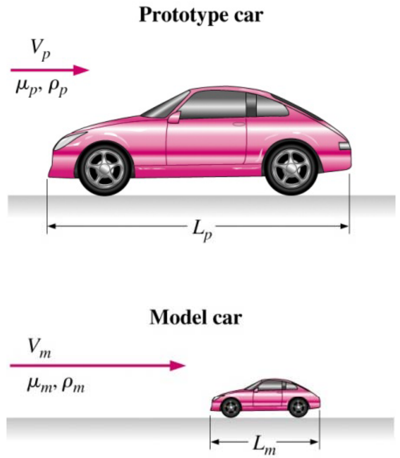


FIGURE 7–16
Kinematic similarity is achieved when, at all locations, the velocity in the model flow is proportional to that at corresponding locations in the prototype flow, and points in the same direction.

7-3 Dimensional Analysis and Similarity (4)

- Complete similarity is ensured if all independent Π groups are the same between model and prototype.
- What is Π?
  - We let uppercase Greek letter Π denote a nondimensional parameter, e.g.,Reynolds number *Re*, Froude number *Fr,* Drag coefficient, C_D, etc.



- Consider automobile experiment
- Drag force is $F = f(V, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to

$$\Pi_1 = f(\Pi_2) \rightarrow C_D = f(Re)$$

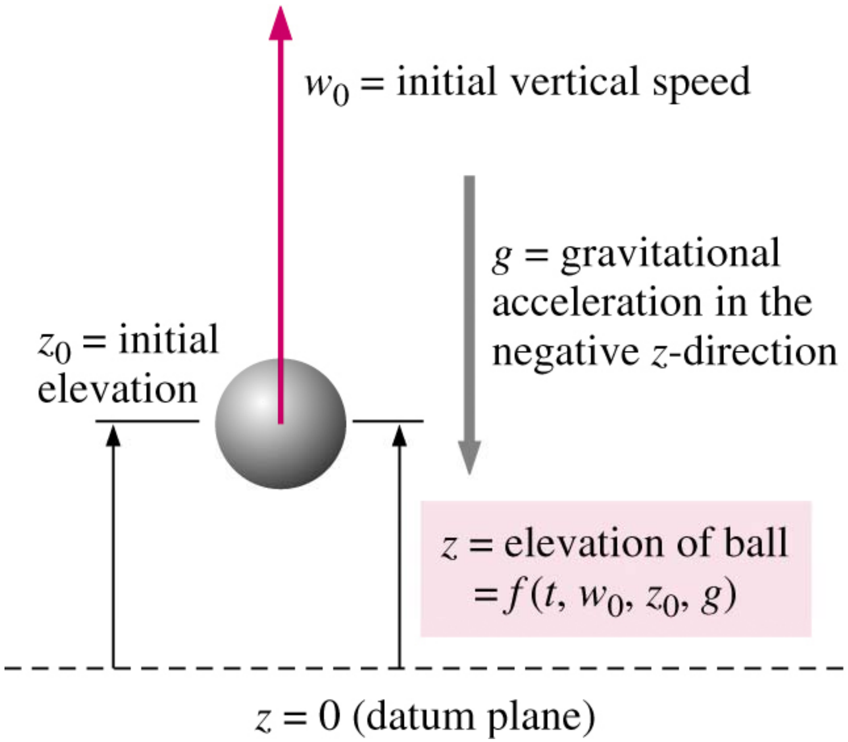
7-4 Method of Repeating Variables (1)

- Nondimensional parameters Π can be generated by several methods.
- We will use the **Method of Repeating Variables**
- Six steps
  1. List the parameters in the problem and count their total number *n*.
  2. List the primary dimensions of each of the *n* parameters
  3. Set the *reduction j* as the number of primary dimensions. Calculate *k*, the expected number of Π's, $k = n - j$.
  4. Choose *j repeating parameters.*
  5. Construct the *k* Π's, and manipulate as necessary.
  6. Write the final functional relationship and check algebra.

7-4 Method of Repeating Variables (1)

Example

Ball Falling in a Vacuum



- Step 1: List relevant parameters. $z=f(t,w_0,z_0,g) \Rightarrow n=5$
- Step 2: Primary dimensions of each parameter

$$\begin{array}{ccccc} z & t & w_0 & z_0 & g \\ \{L^1\} & \{t^1\} & \{L^1t^{-1}\} & \{L^1\} & \{L^1t^{-2}\} \end{array}$$

- Step 3: As a first guess, reduction j is set to 2 which is the number of primary dimensions (L and t). Number of expected Π's is $k=n-j=5-2=3$
- Step 4: Choose repeating variables w_0 and z_0

7-4 Method of Repeating Variables (2)

Guidelines for choosing *Repeating parameters*

1. Never pick the dependent variable. Otherwise, it may appear in all the Π's.
2. Chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
3. Chosen repeating parameters must represent *all* the primary dimensions.
4. Never pick parameters that are already dimensionless.
5. Never pick two parameters with the same dimensions or with dimensions that differ by only an exponent.
6. Choose dimensional constants over dimensional variables so that only one Π contains the dimensional variable.
7. Pick common parameters since they may appear in each of the Π's.
8. Pick simple parameters over complex parameters.

7-4 Method of Repeating Variables (3)

Example, continued

- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's.
- $\Pi_1 = zw_0^{a1}z_0^{b1}$
  - *a1* and *b1* are constant exponents which must be determined.
  - Use the primary dimensions identified in Step 2 and solve for *a1* and *b1*.

$$\{\Pi_1\} = \{L^0 t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1 (L^1 t^{-1})^{a_1} L^{b_1}\}$$

  - Time equation: $\{t^0\} = \{t^{-a_1}\} \rightarrow 0 = -a_1 \rightarrow a_1 = 0$
  - Length equation:

$$\{L^0\} = \{L^1 L^{a_1} L^{b_1}\} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1$$

  - This results in $\Pi_1 = z w_0^0 z_0^{-1} = \frac{z}{z_0}$

7-4 Method of Repeating Variables (4)

- Step 5: continued
  - Repeat process for Π_2 by combining repeating parameters with *t*
  - $\Pi_2 = t w_0^{a2} z_0^{b2}$

$$\{\Pi_2\} = \{L^0 t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t^1 (L^1 t^{-1})^{a_2} L^{b_2}\}$$

  - Time equation:

$$\{t^0\} = \{t^1 t^{-a_2}\} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1$$

  - Length equation:

$$\{L^0\} = \{L^{a_2} L^{b_2}\} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1$$

  - This results in

$$\Pi_2 = t w_0^1 z_0^{-1} = \frac{w_0 t}{z_0}$$

7-4 Method of Repeating Variables (5)

- Step 5: continued
  - Repeat process for Π_3 by combining repeating parameters with *g*
  - $\Pi_3 = g w_0^{a3} z_0^{b3}$

$$\{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$$

  - Time equation:

$$\{t^0\} = \{t^{-2} t^{-a_3}\} \rightarrow 0 = -2 - a_3 \rightarrow a_3 = -2$$

  - Length equation:

$$\{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \rightarrow 0 = 1 + a_3 + b_3 \rightarrow b_3 = -1 - a_3 \rightarrow b_3 = 1$$

  - This results in

$$\Pi_3 = g w_0^{-2} z_0^1 = \frac{g z_0}{w_0^2}$$

$$\Pi_{3,modified} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = Fr$$

7-4 Method of Repeating Variables (6)

- Step 6:
  - Double check that the Π's are dimensionless.
  - Write the functional relationship between Π's

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{g z_0}}\right)$$

  - Or, in terms of nondimensional variables

$$z^* = f(t^*, Fr)$$

- Overall conclusion: Method of repeating variables properly predicts the functional relationship between dimensionless groups.
- However, the method cannot predict the exact mathematical form of the equation.

7-4 Method of Repeating Variables (7)

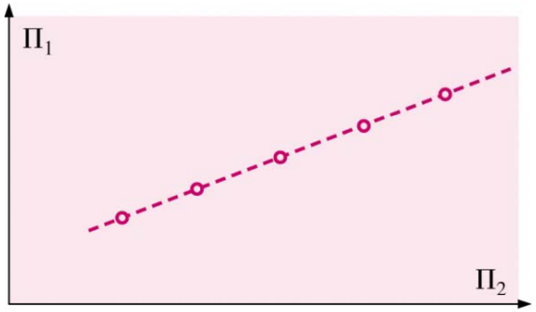
TABLE 7–5 (Continued)

Name	Definition	Ratio of Significance
Mach number	$\text{Ma (sometimes } M) = \frac{V}{c}$	$\frac{\text{Flow speed}}{\text{Speed of sound}}$
Nusselt number	$\text{Nu} = \frac{Lh}{k}$	$\frac{\text{Convection heat transfer}}{\text{Conduction heat transfer}}$
Peclet number	$\text{Pe} = \frac{\rho L V c_p}{k} = \frac{LV}{\alpha}$	$\frac{\text{Bulk heat transfer}}{\text{Conduction heat transfer}}$
Power number	$N_P = \frac{\dot{W}}{\rho D^5 \omega^3}$	$\frac{\text{Power}}{\text{Rotational inertia}}$
Prandtl number	$\text{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$	$\frac{\text{Viscous diffusion}}{\text{Thermal diffusion}}$
Pressure coefficient	$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V^2}$	$\frac{\text{Static pressure difference}}{\text{Dynamic pressure}}$
Rayleigh number	$\text{Ra} = \frac{g\beta \lvert\Delta T\rvert L^3 \rho^2 c_p}{k\mu}$	$\frac{\text{Buoyancy force}}{\text{Viscous force}}$
Reynolds number	$\text{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$	$\frac{\text{Inertial force}}{\text{Viscous force}}$
Richardson number	$\text{Ri} = \frac{L^5 g \,\Delta\rho}{\rho \dot{V}^2}$	$\frac{\text{Buoyancy force}}{\text{Inertial force}}$
Schmidt number	$\text{Sc} = \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$	$\frac{\text{Viscous diffusion}}{\text{Species diffusion}}$
Sherwood number	$\text{Sh} = \frac{VL}{D_{AB}}$	$\frac{\text{Overall mass diffusion}}{\text{Species diffusion}}$
Specific heat ratio	$k \text{ (sometimes } \gamma) = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$
Stanton number	$\text{St} = \frac{h}{\rho c_p V}$	$\frac{\text{Heat transfer}}{\text{Thermal capacity}}$
Stokes number	$\text{Stk (sometimes St)} = \frac{\rho_p D_p^2 V}{18\mu L}$	$\frac{\text{Particle relaxation time}}{\text{Characteristic flow time}}$
Strouhal number	$\text{St (sometimes S or Sr)} = \frac{fL}{V}$	$\frac{\text{Characteristic flow time}}{\text{Period of oscillation}}$
Weber number	$\text{We} = \frac{\rho V^2 L}{\sigma_s}$	$\frac{\text{Inertial force}}{\text{Surface tension force}}$

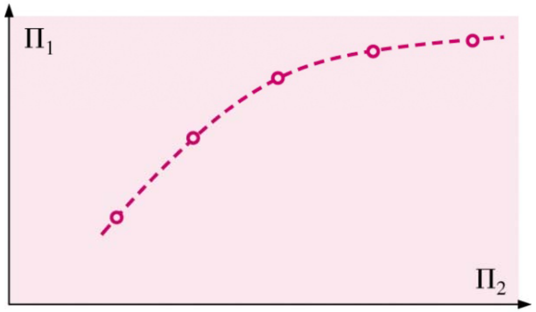
* A is a characteristic area, D is a characteristic diameter, f is a characteristic frequency (Hz), L is a characteristic length, t is a characteristic time, T is a characteristic (absolute) temperature, V is a characteristic velocity, W is a characteristic width, $\dot{W}$ is a characteristic power, ω is a characteristic angular velocity (rad/s). Other parameters and fluid properties in these Π's include: c = speed of sound, c_p, c_v = specific heats, D_p = particle diameter, D_{AB} = species diffusion coefficient, h = convective heat transfer coefficient, h_{fg} = latent heat of evaporation, k = thermal conductivity, P = pressure, T_{sat} = saturation temperature, $\dot{V}$ = volume flow rate, α = thermal diffusivity, β = coefficient of thermal expansion, λ = mean free path length, μ = viscosity, ν = kinematic viscosity, ρ = fluid density, ρ_f = liquid density, ρ_p = particle density, ρ_s = solid density, ρ_v = vapor density, σ_s = surface tension, and τ_w = shear stress along a wall.

7-5 Experimental Testing and Incomplete Similarity (1)

- One of the most useful applications of dimensional analysis is in designing physical and/or numerical experiments, and in reporting the results.
- Setup of an experiment and correlation of data.



(a)



(b)

- Consider a problem with 5 parameters: one dependent and 4 independent.
- Full test matrix with 5 data points for each independent parameter would require 5^4=625 experiments!!
- If we can reduce to 2 Π's, the number of independent parameters is reduced from 4 to 1, which results in 5^1=5 experiments vs. 625!!

7-5 Experimental Testing and Incomplete Similarity (2)

Wanapum Dam on Columbia River

Physical Model at
Iowa Institute of Hydraulic Research

- Flows with free surfaces present unique challenges in achieving complete dynamic similarity.
- For hydraulics applications, depth is very small in comparison to horizontal dimensions. If geometric similarity is used, the model depth would be so small that other issues would arise
  - Surface tension effects (Weber number) would become important.
  - Data collection becomes difficult.
- Distorted models are therefore employed, which requires empirical corrections/correlations to extrapolate model data to full scale.

7-5 Experimental Testing and Incomplete Similarity (3)

DDG-51 Destroyer

1/20th scale model

- For ship hydrodynamics, Fr similarity is maintained while Re is allowed to be different.
- Why? Look at complete similarity:

$$Re_p = \frac{V_p L_p}{\nu_p} = Re_m = \frac{V_m L_m}{\nu_m} \rightarrow \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m}$$

$$Fr_p = \frac{V_p}{\sqrt{gL_p}} = Fr_m = \frac{V_m}{\sqrt{gL_m}} \rightarrow \frac{L_m}{L_p} = \left(\frac{V_p}{V_m}\right)^2$$

- To match both Re and *Fr*, viscosity in the model test is a function of scale ratio! ***This is not feasible.***

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$