



Chapter 6: Momentum Analysis of Flow Systems



6-1 Introduction

**6-2 Newton's Law and Conservation
of Momentum**

6-3 Choosing a Control Volume

6-4 Forces Acting on a Control Volume

6-5 Linear Momentum Equation

6-6 Angular Momentum

6-7 The Second Law of Thermodynamics

6-1 Introduction

- Fluid flow problems can be analyzed using one of three basic approaches: differential, experimental, and integral (or control volume).
- In Chap. 5, control volume forms of the mass and energy equation were developed and used.
- In this chapter, we complete control volume analysis by presenting the integral momentum equation.
 - Review Newton's laws and conservation relations for momentum.
 - Use RTT to develop linear and angular momentum equations for control volumes.
 - Use these equations to determine forces and torques acting on the CV.

6-1

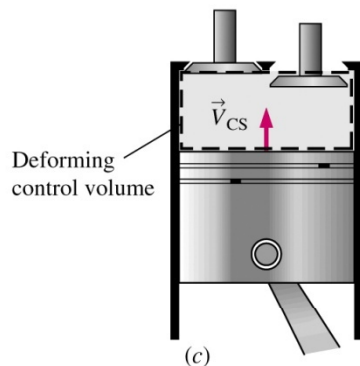
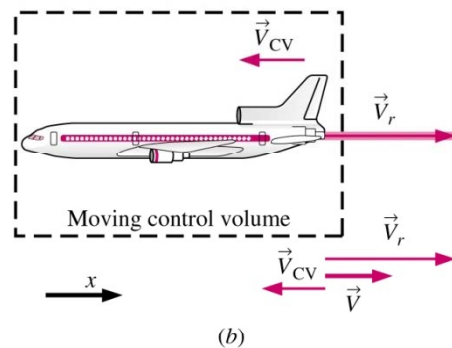
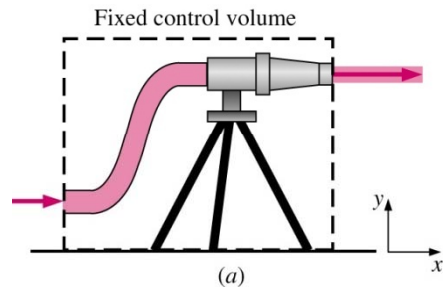
6-2 Newton's Laws and Conservation of Momentum (1)

- Newton's laws are relations between motions of bodies and the forces acting on them.
 - **First law:** a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
 - **Second law:** the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\vec{F} = m\vec{a} = m \frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt}$$

- **Third law:** when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

6-3 Choosing a Control Volume



- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
 - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
 - Clearly identify all fluxes crossing the CS.
 - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.

- For moving CV, use relative velocity,

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

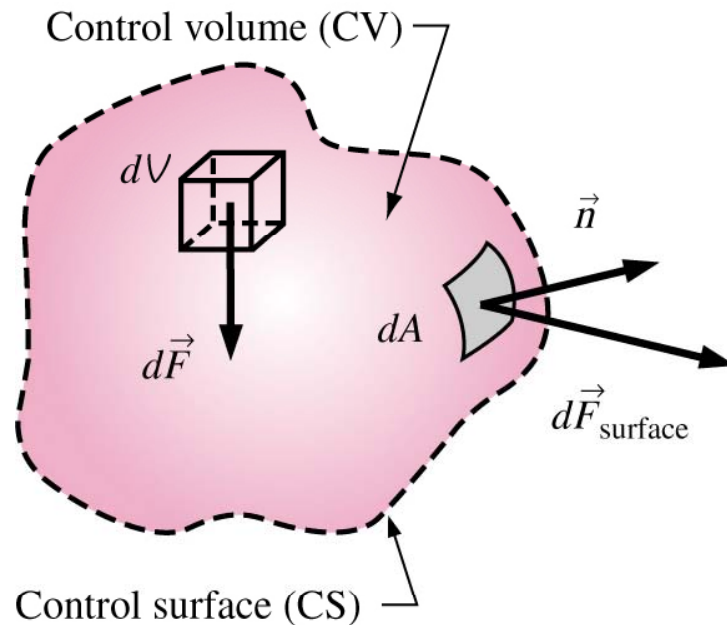
- For deforming CV, use relative velocity all deforming control surfaces,

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

6-3

6-4 Forces Acting on a CV (1)

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).

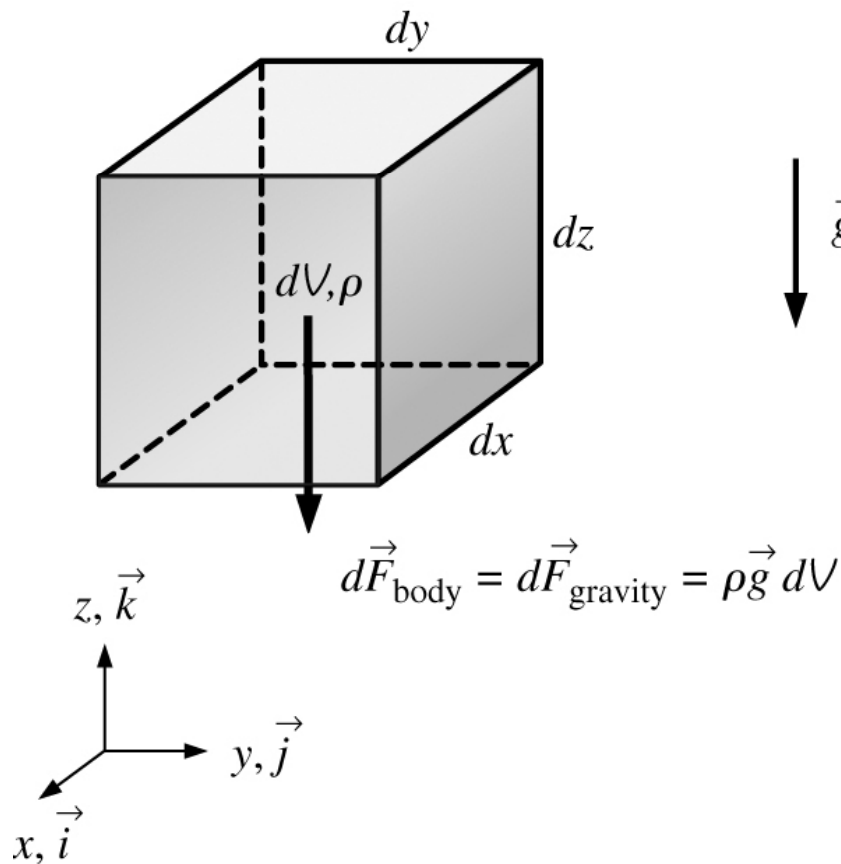


- Body forces act on each volumetric portion dV of the CV.
- Surface forces act on each portion dA of the CS.

6-4

6-4 Forces Acting on a CV (2)

Body Forces



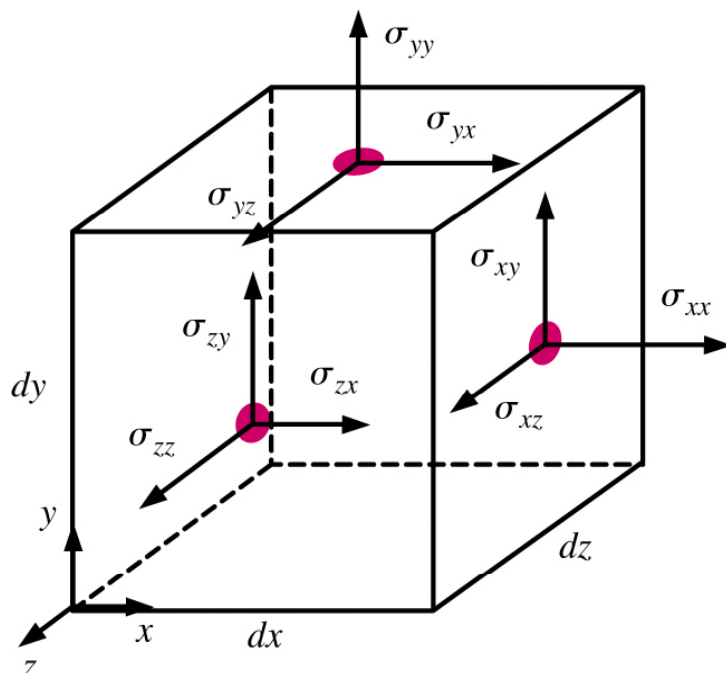
- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The differential body force
$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho \vec{g} dV$$
- Typical convention is that $\vec{g}$ acts in the negative z-direction,
$$\vec{g} = -g\vec{k}$$
- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho \vec{g} dV = m_{CV} \vec{g}$$

6-5

6-4 Forces Acting on a CV (3)

Surface Forces



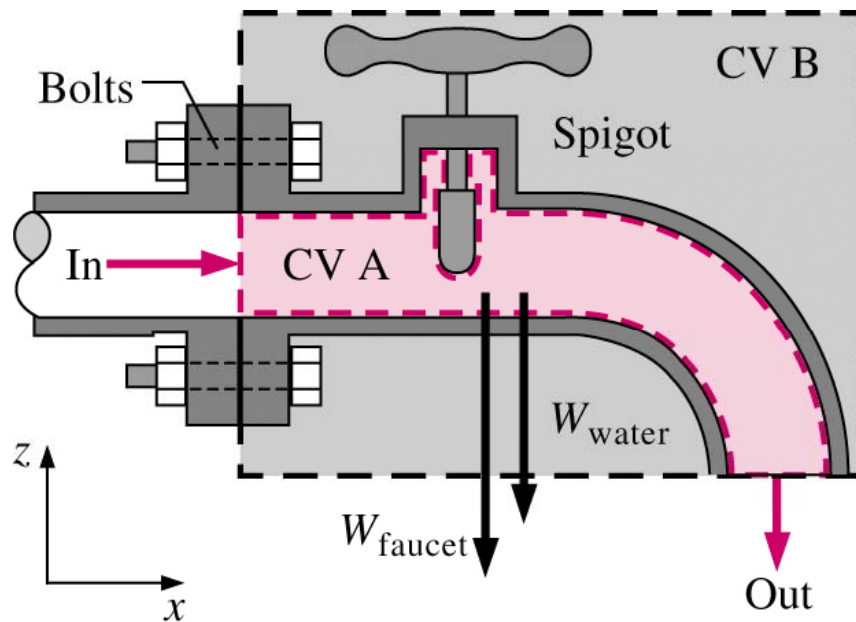
- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components σ_{xx} , σ_{yy} , σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy} , σ_{xz} , etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

6-6

6-4 Forces Acting on a CV (4)



- Surface integrals are cumbersome.
- Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, pressure, and reaction forces.
- Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

6-7

6-5 Linear Momentum Equation (1)

- Newton's second law for a system of mass m subjected to a force $\vec{F}$ is expressed as

$$\vec{F} = m\vec{a} = m \frac{d\vec{V}}{dt} = \frac{d}{dt} (m\vec{V})$$

- Use RTT with $b = \vec{V}$ and $B = m\vec{V}$ to shift from system formulation of the control volume formulation

$$\frac{d(m\vec{V})_{sys}}{dt} = \frac{d}{dt} \int_{CV} \rho \vec{V} d\mathcal{V} + \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$$

$$\sum \vec{F} = \frac{d}{dt} \int_{CV} \rho \vec{V} d\mathcal{V} + \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$$

6-8

6-5 Linear Momentum Equation (2)

- Steady Flow $\sum \vec{F} = \int_{CS} \rho \vec{V} (\vec{V}_r \cdot \vec{n}) dA$
- Average velocities $\dot{m} = \int_{A_C} \rho (\vec{V} \cdot \vec{n}) dA_c = \rho V_{avg} A_c$
- Approximate momentum flow rate

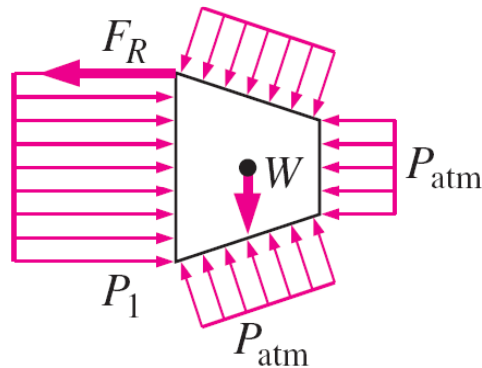
$$\int_{CS} \rho \vec{V} (\vec{V} \cdot \vec{n}) dA \approx \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$

- To account for error, use momentum-flux correction factor β

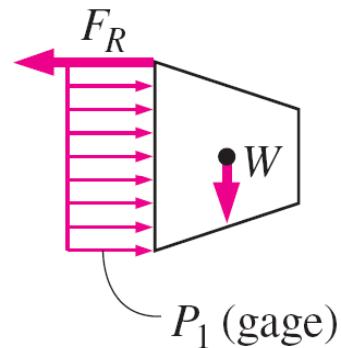
$$\sum \vec{F} = \frac{d}{dt} \int \rho \vec{V} d\mathcal{V} + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$
$$\beta = \frac{1}{A_c} \int_{A_C} \left(\frac{V}{V_{avg}} \right)^2 dA_c$$

6-9

6-5 Linear Momentum Equation (3)



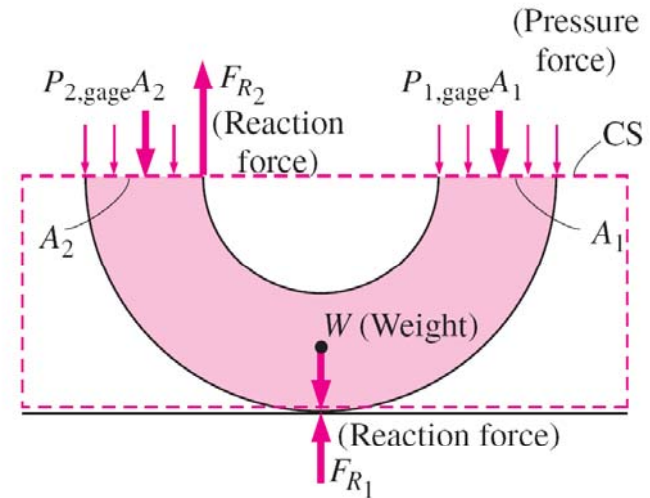
With atmospheric pressure considered



With atmospheric pressure cancelled out

FIGURE 6-9

Atmospheric pressure acts in all directions, and thus it can be ignored when performing force balances since its effect cancels out in every direction.



An 180° elbow supported by the ground

FIGURE 6-12

In most flow systems, the force $\vec{F}$ consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

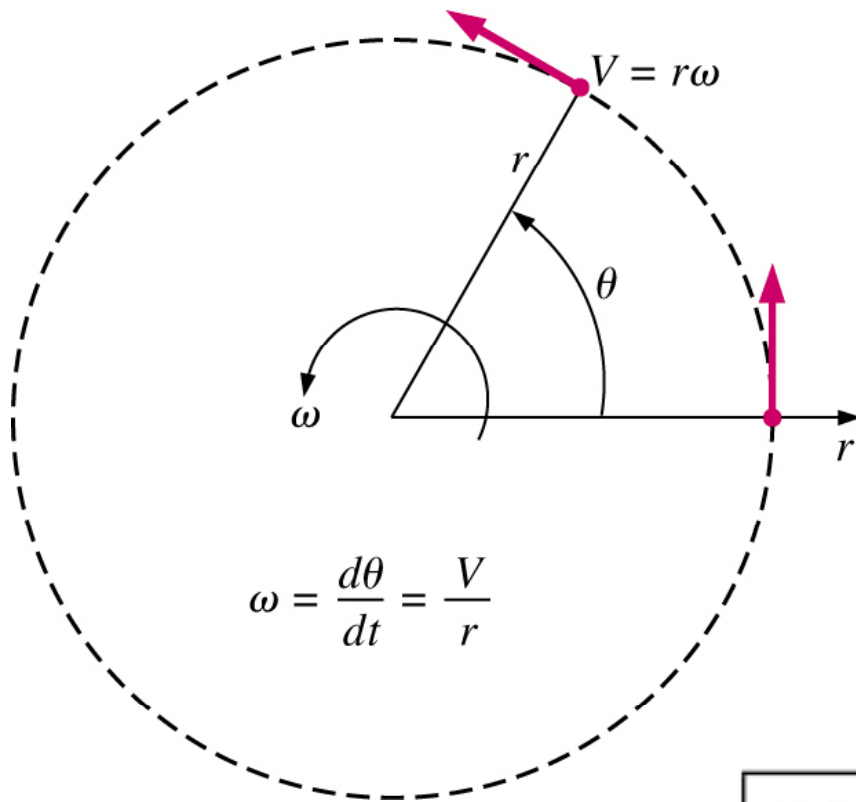
6-10

6-6 Angular Momentum (1)

- Motion of a rigid body can be considered to be the combination of
 - the translational motion of its center of mass (U_x, U_y, U_z)
 - the rotational motion about its center of mass ($\omega_x, \omega_y, \omega_z$)
- Translational motion can be analyzed with linear momentum equation.
- Rotational motion is analyzed with angular momentum equation.
- Together, the body motion can be described as a 6-degree-of-freedom (6DOF) system.

6-11

6-6 Angular Momentum (2)



Angular velocity ω is the angular distance θ traveled per unit time, and angular acceleration α is the rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(lr)}{dt} = \frac{1}{r} \frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r} \frac{dV}{dt} = \frac{a_t}{r}$$

$$V = r\omega \text{ and } a_t = r\alpha$$

6-12

6-6 Angular Momentum (3)

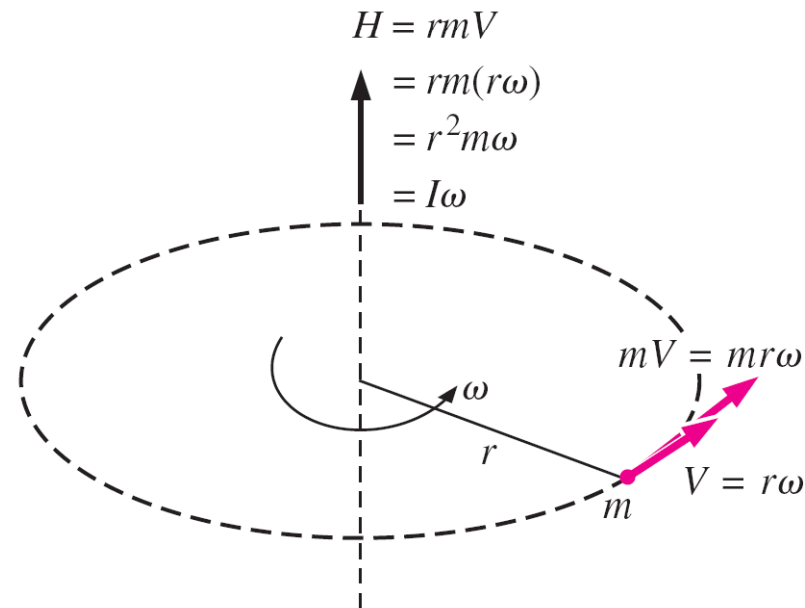


FIGURE 6-29

Angular momentum of point mass m
rotating at angular velocity ω at
distance r from the axis of rotation.

6-6 Angular Momentum (4)

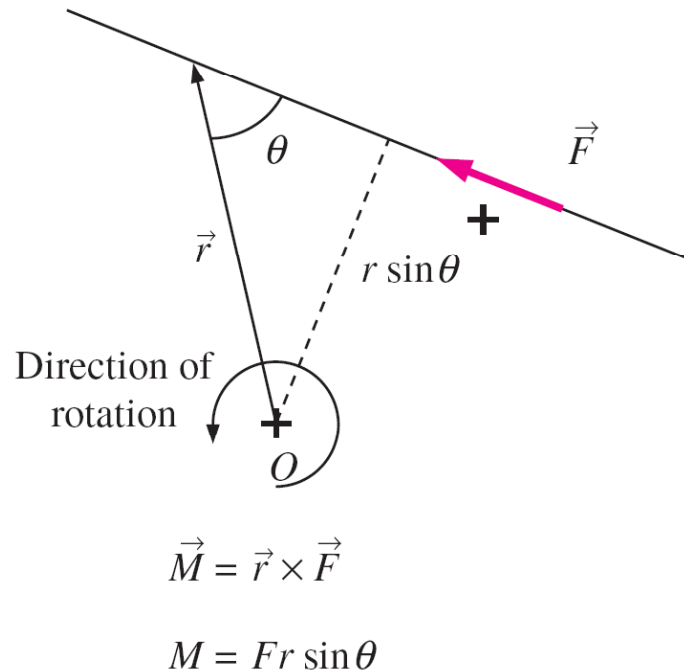


FIGURE 6-31

The moment of a force $\vec{F}$ about a point O is the vector product of the position vector $\vec{r}$ and $\vec{F}$.

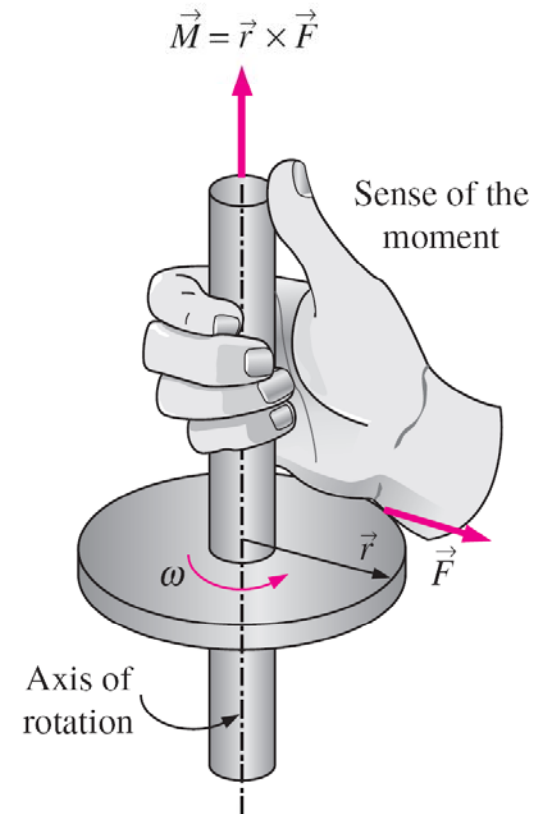


FIGURE 6-32

The determination of the direction of the moment by the right-hand rule.

6-14

6-6 Angular Momentum (5)

- Moment of a force: $\vec{M} = \vec{r} \times \vec{F}$
- Moment of momentum: $\vec{H} = \vec{r} \times m\vec{V}$
- For a system: $\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{V})\rho d\mathcal{V}$
 $\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt} \int_{sys} (\vec{r} \times \vec{V})\rho d\mathcal{V}$
- Therefore, the angular momentum equation can be written as: $\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt}$
- To derive angular momentum for a CV, use RTT with $B = \vec{H}$ and $\beta = \vec{r} \times \vec{V}$

6-15

6-6 Angular Momentum (6)

■ General form

$$\sum \vec{M} = \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho d\mathcal{V} + \int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V}_r \cdot \vec{n}) dA$$

■ Approximate form using average properties at inlets and outlets

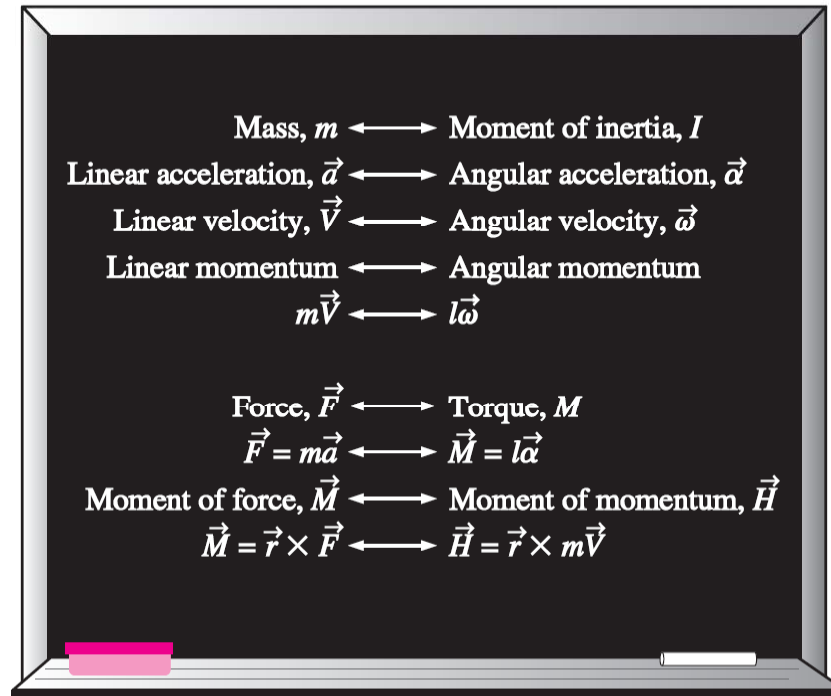
$$\sum \vec{M} = \frac{d}{dt} \int_{CV} (\vec{r} \times \vec{V}) \rho d\mathcal{V} + \sum_{out} \vec{r} \times \dot{m} \vec{V} - \sum_{in} \vec{r} \times \dot{m} \vec{V}$$

■ Steady flow

$$\sum \vec{M} = + \sum_{out} \vec{r} \times \dot{m} \vec{V} - \sum_{in} \vec{r} \times \dot{m} \vec{V}$$

6-16

6-6 Angular Momentum (7)



Mass, m	↔	Moment of inertia, I
Linear acceleration, $\vec{a}$	↔	Angular acceleration, $\vec{\alpha}$
Linear velocity, $\vec{V}$	↔	Angular velocity, $\vec{\omega}$
Linear momentum	↔	Angular momentum
$m\vec{V}$	↔	$I\vec{\omega}$
Force, $\vec{F}$	↔	Torque, M
$\vec{F} = m\vec{a}$	↔	$\vec{M} = I\vec{\alpha}$
Moment of force, $\vec{M}$	↔	Moment of momentum, $\vec{H}$
$\vec{M} = \vec{r} \times \vec{F}$	↔	$\vec{H} = \vec{r} \times m\vec{V}$

FIGURE 6-28

Analogy between corresponding
linear and angular quantities.

6-7 The Second Law of Thermodynamics

$$\left(\frac{dS}{dt} \right)_{\text{system}} \geq \frac{1}{T} \dot{Q}$$

$$S_{\text{system}} = \int_{M(\text{system})} s \, dm = \int_{\forall(\text{system})} s \, \rho \, d\forall$$

$$\left(\frac{dS}{dt} \right)_{\text{system}} = \frac{\partial}{\partial t} \int_{\text{CV}} s \, \rho \, d\forall + \int_{\text{CS}} s \, \rho \, \vec{V} \cdot d\vec{A}$$

$$\frac{\partial}{\partial t} \int_{\text{CV}} s \, \rho \, d\forall + \int_{\text{CS}} s \, \rho \, \vec{V} \cdot d\vec{A} \geq \int_{\text{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) dA$$

6-18