Chapter 5: Mass, Bernoulli, and Energy Equations

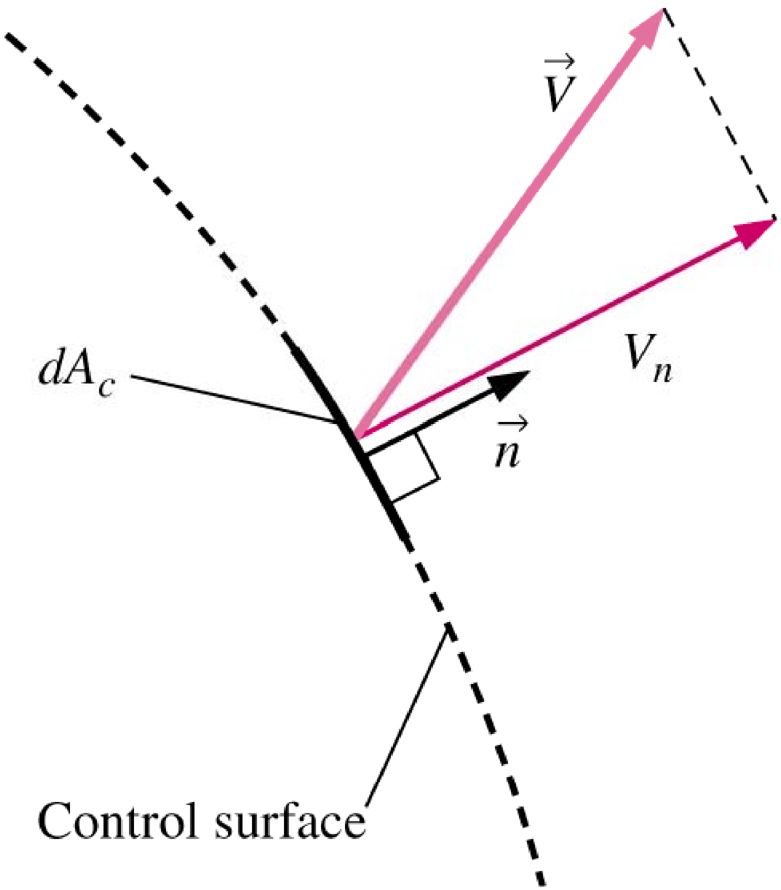
5-1 Introduction

- This chapter deals with 3 equations commonly used in fluid mechanics
  - *The mass equation* is an expression of the conservation of mass principle.
  - *The Bernoulli equation* is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other.
  - *The energy equation* is a statement of the conservation of energy principle.

5-2 Conservation of Mass (1)

- Conservation of mass principle is one of the most fundamental principles in nature.
- Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process.
- For *closed systems* mass conservation is implicit since the mass of the system remains constant during a process.
- For *control volumes*, mass can cross the boundaries which means that we must keep track of the amount of mass entering and leaving the control volume.

5-2 Conservation of Mass (2)

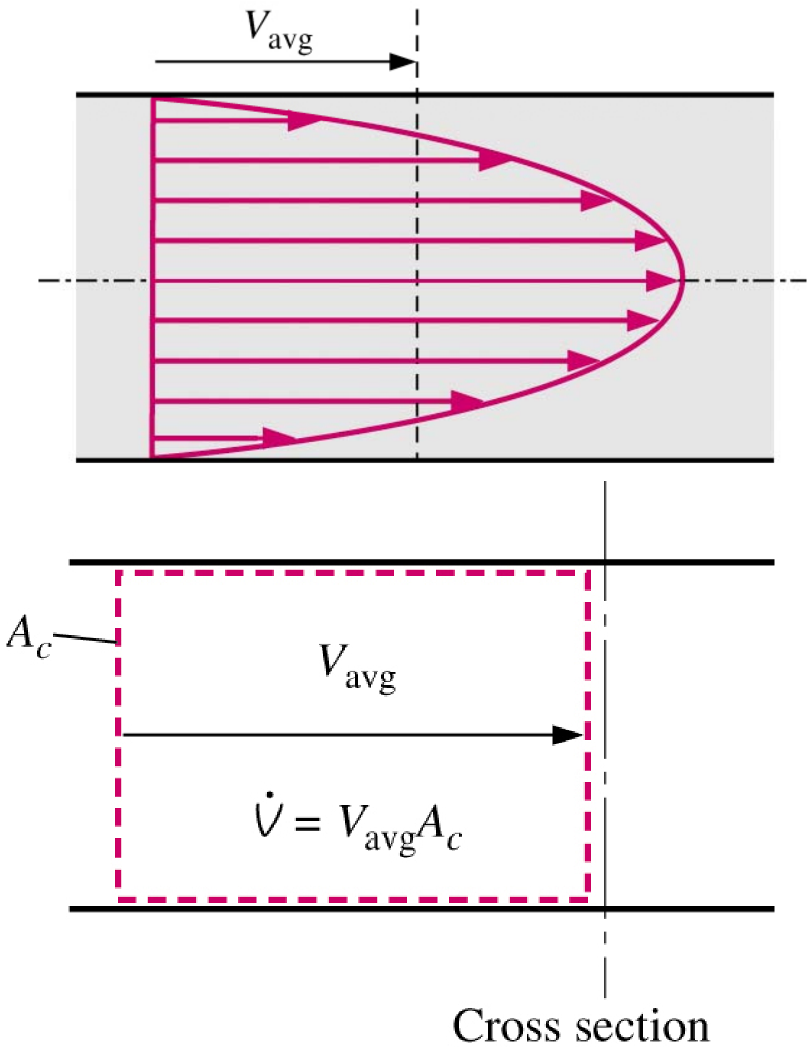


- The amount of mass flowing through a control surface per unit time is called the **mass flow rate** and is denoted $\dot{m}$
- The dot over a symbol is used to indicate *time rate of change*.
- Flow rate across the entire cross-sectional area of a pipe or duct is obtained by integration

$$\dot{m} = \int_{A_c} \delta m = \int_{A_c} \rho V_n dA_c$$

- While this expression for $\dot{m}$ is exact, it is not always convenient for engineering analyses.

5-2 Conservation of Mass (3)



- Integral in $\dot{m}$ can be replaced with average values of ρ and V_n

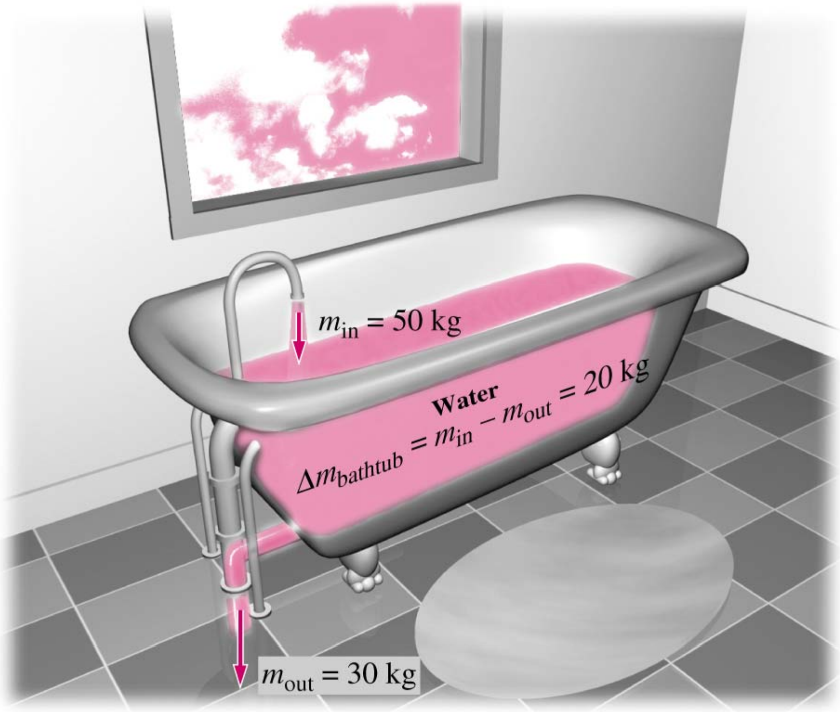
$$V_{avg} = \frac{1}{A_c} \int_{A_c} V_n \, dA_c$$

- For many flows variation of ρ is very small: $\dot{m} = \rho V_{avg} A_c$
- Volume flow rate $\dot{V}$ is given by

$$\dot{V} = \int_{A_c} V_n \, dA_c = V_{avg} A_c = V A_c$$

- Note: many textbooks use Q instead of $\dot{V}$ for volume flow rate.
- Mass and volume flow rates are related by $\dot{m} = \rho \dot{V}$

5-2 Conservation of Mass (4)

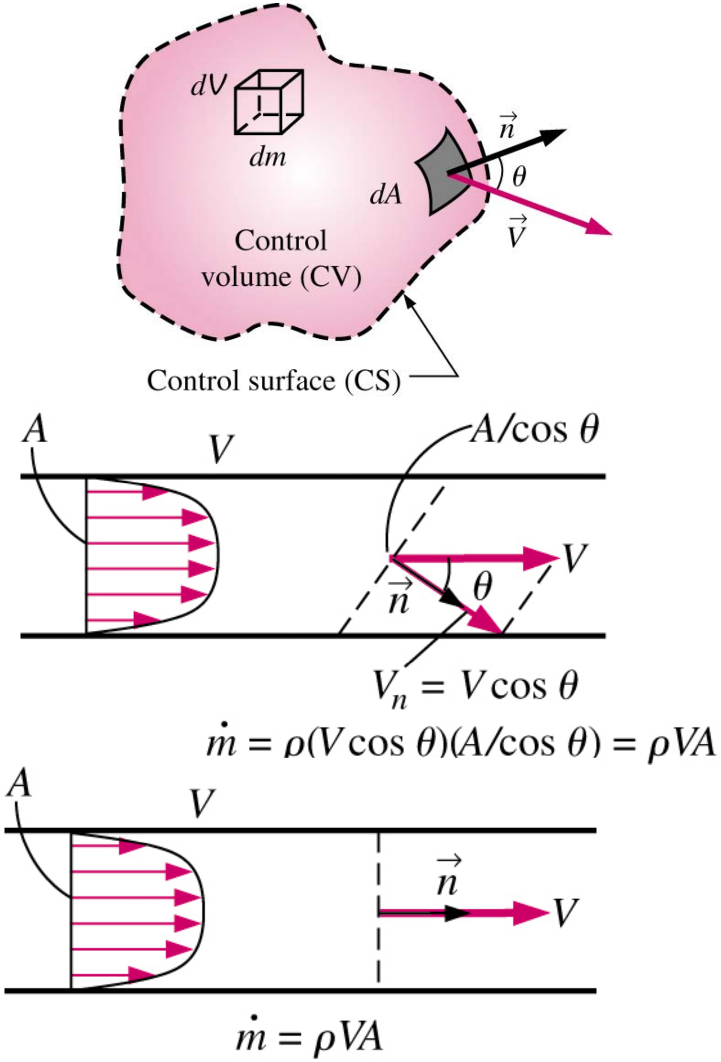


- The **conservation of mass principle** can be expressed as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

- Where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the CV, and dm_{CV}/dt is the rate of change of mass within the CV.

5-2 Conservation of Mass (5)



- For CV of arbitrary shape,
  - rate of change of mass within the CV

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho dV\!\!\!\!-$$

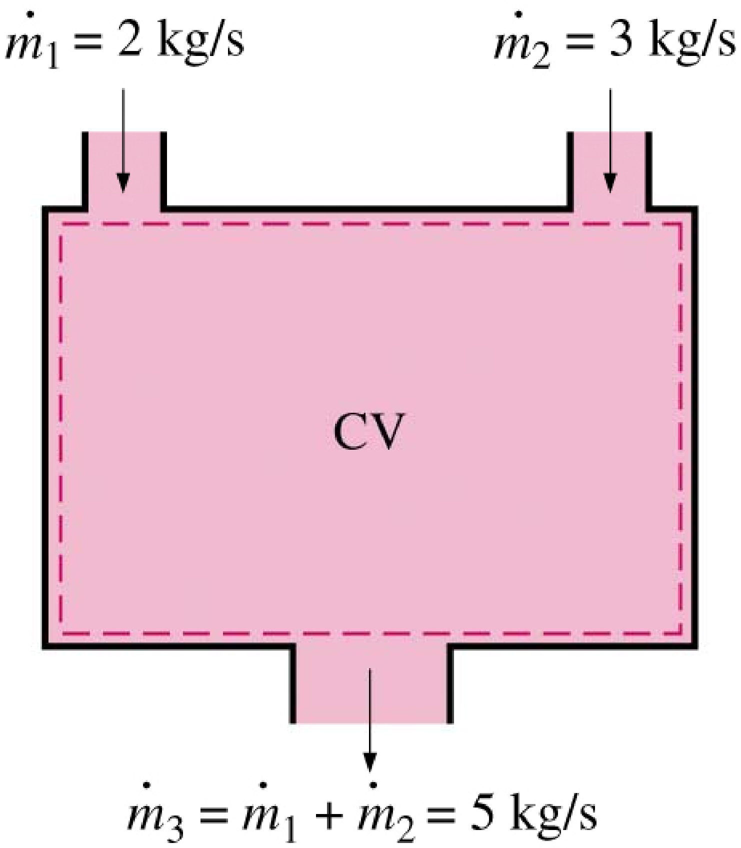
  - net mass flow rate

$$\dot{m}_{net} = \int_{CS} \delta\dot{m} = \int_{CS} \rho V_n dA = \int_{CS} \rho\left(\vec{V}\cdot\vec{n}\right) dA$$

- Therefore, general conservation of mass for a fixed CV is:

$$\frac{d}{dt}\int_{CV} \rho dV\!\!\!\!- + \int_{CS} \rho\left(\vec{V}\cdot\vec{n}\right) dA = 0$$

5-2 Conservation of Mass (6)



- For steady flow, the total amount of mass contained in CV is constant.
- Total amount of mass entering must be equal to total amount of mass leaving

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

- For incompressible flows,

$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

5-3 Mechanical Energy (1)

- **Mechanical energy** can be defined as *the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine.*
- Flow P/ρ, kinetic V^2/g, and potential gz energy are the forms of mechanical energy $e_{mech} = P/\rho + V^2/g + gz$
- Mechanical energy change of a fluid during incompressible flow becomes

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

- In the absence of loses, Δe_{mech} represents the work supplied to the fluid ($\Delta e_{mech}>0$) or extracted from the fluid ($\Delta e_{mech}<0$).

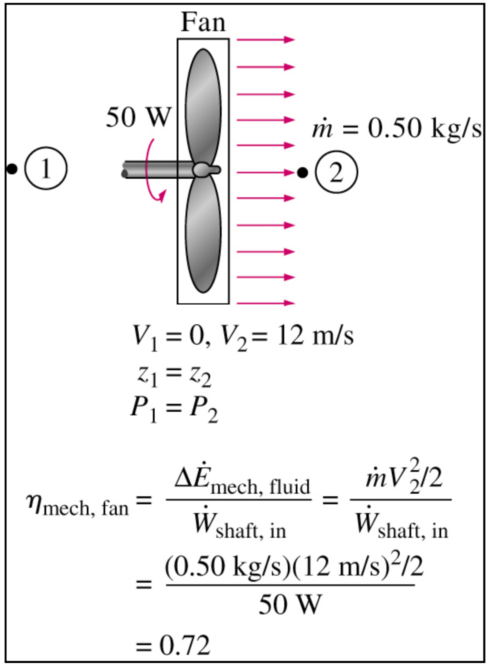
5-3 Mechanical Energy (2)

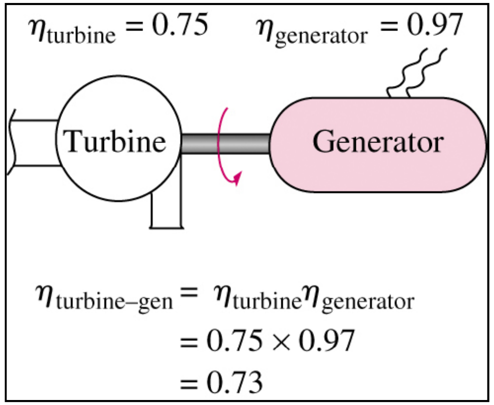
- Transfer of e_{mech} is usually accomplished by a rotating shaft: *shaft work*
- Pump, fan, propulsion: receives shaft work (e.g., from an electric motor) and transfers it to the fluid as mechanical energy
- Turbine: converts e_{mech} of a fluid to shaft work.
- In the absence of irreversibilities (e.g., friction), **mechanical efficiency** of a device or process can be defined as

$$\eta_{mech} = \frac{E_{mech,out}}{E_{mech,in}} = 1 - \frac{E_{mech,loss}}{E_{mech,in}}$$

- If η_{mech} < 100%, losses have occurred during conversion.

5-3 Mechanical Energy (3)





- In fluid systems, we are usually interested in increasing the pressure, velocity, and/or elevation of a fluid.
- In these cases, efficiency is better defined as the ratio of (supplied or extracted work) vs. rate of increase in mechanical energy

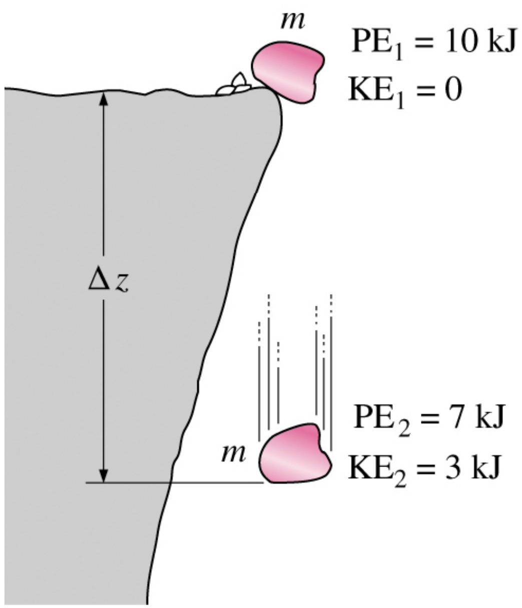
$$\eta_{pump} = \frac{\Delta \dot{E}_{mech,\, fluid}}{\dot{W}_{shaft,in}}$$

$$\eta_{turbine} = \frac{\dot{W}_{shaft,out}}{\left|\Delta \dot{E}_{mech,\, fluid}\right|}$$

- Overall efficiency must include motor or generator efficiency.

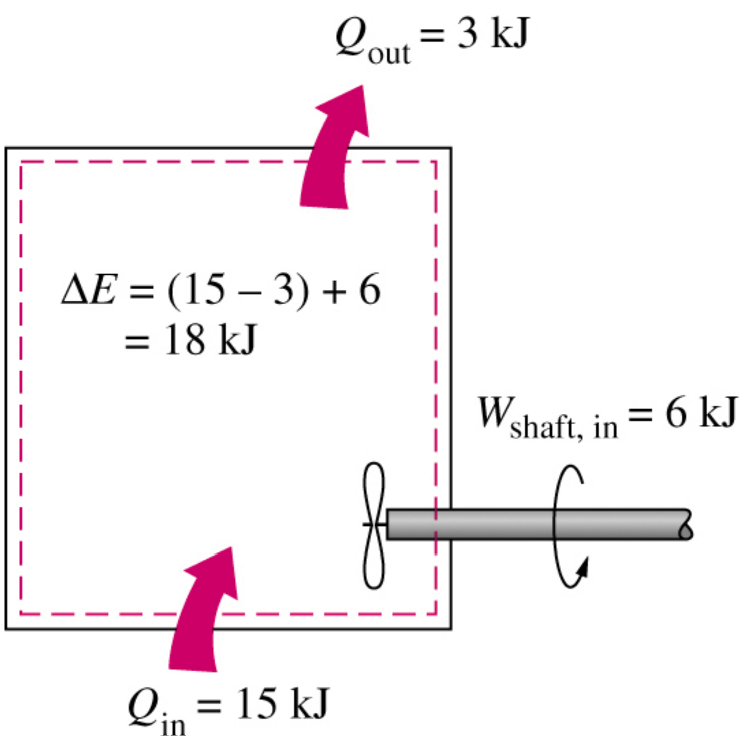
5-4 General Energy Equation (1)

- One of the most fundamental laws in nature is the **1st law of thermodynamics**, which is also known as the **conservation of energy principle**.
- It states that *energy can be neither created nor destroyed during a process; it can only change forms*



- Falling rock, picks up speed as PE is converted to KE.
- If air resistance is neglected, PE + KE = constant

5-4 General Energy Equation (2)



- The energy content of a closed system can be changed by two mechanisms: *heat transfer Q* and *work transfer W*.
- Conservation of energy for a closed system can be expressed in rate form as

$$\dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{dE_{sys}}{dt}$$

- Net rate of heat transfer to the system:

$$\dot{Q}_{net,in} = \dot{Q}_{in} - \dot{Q}_{out}$$

- Net power input to the system:

$$\dot{W}_{net,in} = \dot{W}_{in} - \dot{W}_{out}$$

5-4 General Energy Equation (3)

- Recall general RTT

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, d\mathcal{V} + \int_{CS} \rho b \left(\vec{V}_r \cdot \vec{n}\right) dA$$

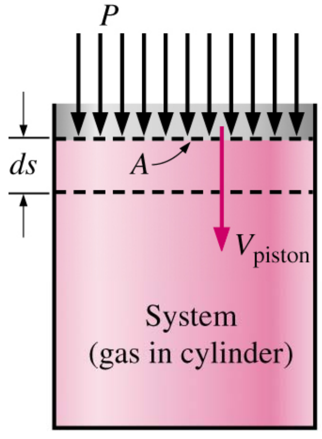
- "Derive" energy equation using *B=E* and *b=e*

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{d}{dt}\int_{CV} \rho e \, d\mathcal{V} + \int_{CS} \rho e \left(\vec{V}_r \cdot \vec{n}\right) dA$$

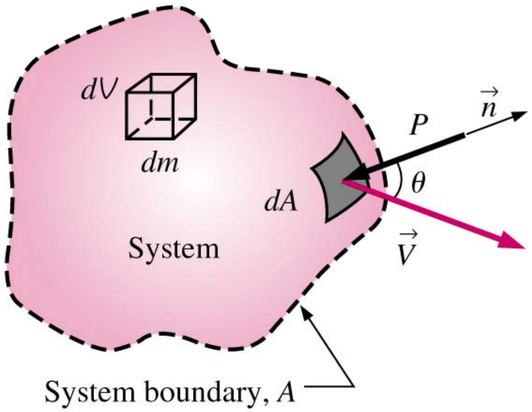
- Break power into rate of shaft and pressure work

$$\dot{W}_{net,in} = \dot{W}_{shaft,net,in} + \dot{W}_{pressure,net,in} = \dot{W}_{shaft,net,in} - \int P\left(\vec{V} \cdot \vec{n}\right) dA$$

5-4 General Energy Equation (4)



(*a*)



(*b*)

- Where does expression for pressure work come from?
- When piston moves down *ds* under the influence of *F=PA*, the work done on the system is $\delta W_{boundary}=PAds$.
- If we divide both sides by *dt*, we have

$$\delta \dot{W}_{pressure} = \delta \dot{W}_{boundary} = PA\frac{ds}{dt} = PAV_{piston}$$

- For generalized control volumes:

$$\delta \dot{W}_{pressure} = -PdAV_n = -PdA\left(\vec{V}\cdot\vec{n}\right)$$

- Note sign conventions:
  - $\vec{n}$ is outward pointing normal
  - Negative sign ensures that work done is positive when is done *on* the system.

5-4 General Energy Equation (5)

- Moving integral for rate of pressure work to RHS of energy equation results in:

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e \, d\mathcal{V} + \int_{CS} \left(\frac{P}{\rho} + e\right) \rho \left(\vec{V}_r \cdot \vec{n}\right) dA$$

- Recall that P/ρ is the **flow work,** which is the work associated with pushing a fluid into or out of a CV per unit mass.

5-4 General Energy Equation (6)

- As with the mass equation, practical analysis is often facilitated as averages across inlets and exits

$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\forall + \sum_{out} \dot{m}\left(\frac{P}{\rho} + e\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + e\right)$$

$$\dot{m} = \int_{A_C} \rho\left(\vec{V}\cdot\vec{n}\right) dA_c$$

- Since $e=u+ke+pe = u+V^2/2+gz$

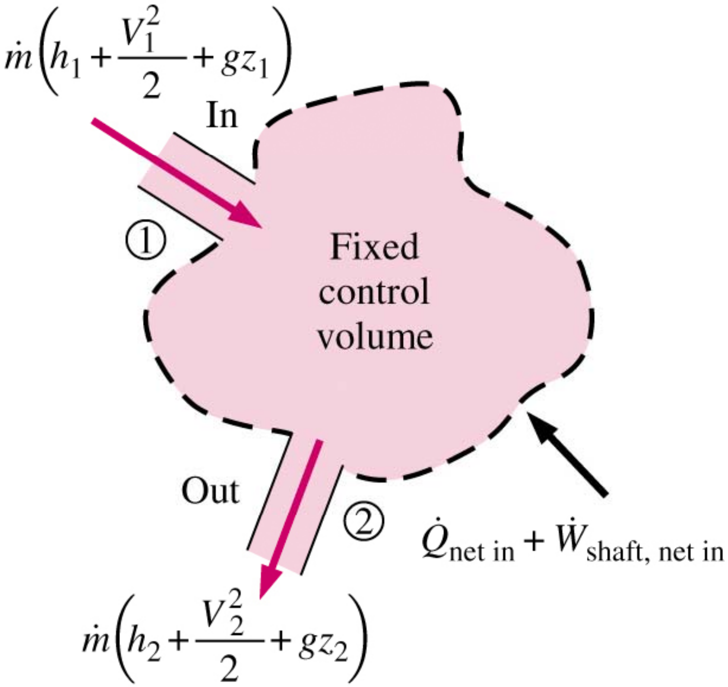
$$\dot{Q}_{net,in} + \dot{W}_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\forall + \sum_{out} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right)$$

5-5 Energy Analysis of Steady Flows (1)

$$Q_{net,in} + W_{shaft,net,in} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

- For steady flow, time rate of change of the energy content of the CV is zero.
- This equation states: *the net rate of energy transfer to a CV by heat and work transfers during steady flow is equal to the difference between the rates of outgoing* and incoming energy flows with mass.

5-5 Energy Analysis of Steady Flows (2)



- For **single-stream devices**, mass flow rate is constant.

$$q_{net,in} + w_{shaft,net,in} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

$$w_{shaft,net,in} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + \left(u_2 - u_1 - q_{net,in}\right)$$

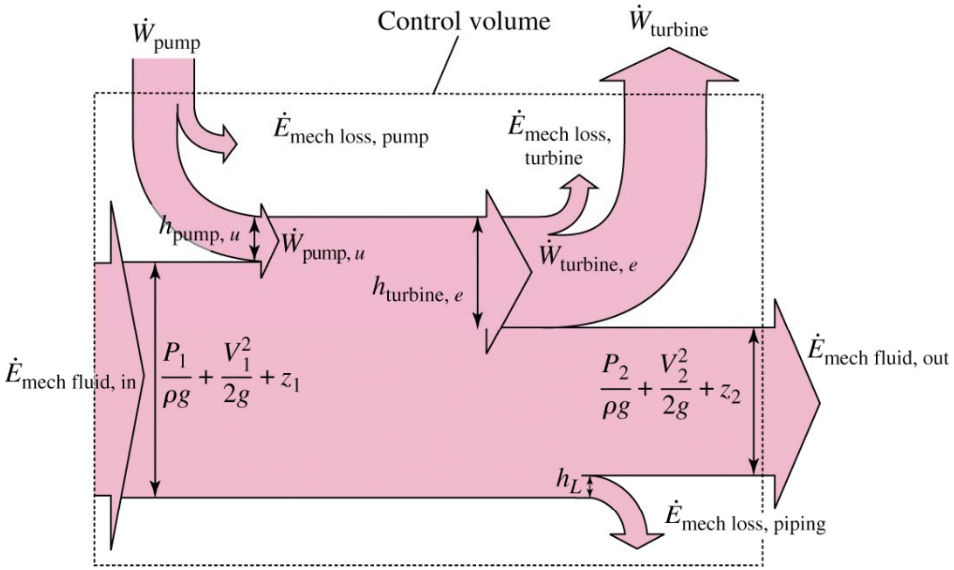
$$\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 + w_{pump} = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + w_{turbine} + e_{mech,loss}$$

5-5 Energy Analysis of Steady Flows (3)

- Divide by *g* to get each term in units of length

$$\frac{P_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1+h_{pump}=\frac{P_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2+h_{turbine}+h_L$$

Magnitude of each term is now expressed as an equivalent column height of fluid, i.e., *Head*



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5-6 The Bernoulli Equation (1)

- If we neglect piping losses, and have a system without pumps or turbines

$$\frac{P_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1=\frac{P_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2$$

- This is the **Bernoulli equation**
- It can also be derived using Newton's second law of motion (see text, p. 187).
- 3 terms correspond to: pressure, velocity, and elevation head.

- Each term has the units of length and represents a certain type of head.
- The elevation term, z, is related to the potential energy of the particle and is called **elevation head**.
- *p*/γ, is the **pressure head** and represents the height of a column of the fluid that is needed to produce the pressure *p*.
- $V^2/2g$,is the **velocity head** and represents the vertical distance for the fluid to fall freely (neglecting friction) if it is to reach velocity *V* from rest.

5-6 The Bernoulli Equation (2)

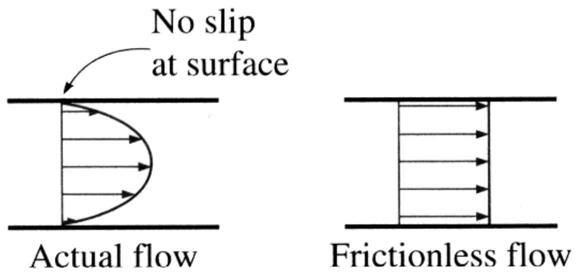
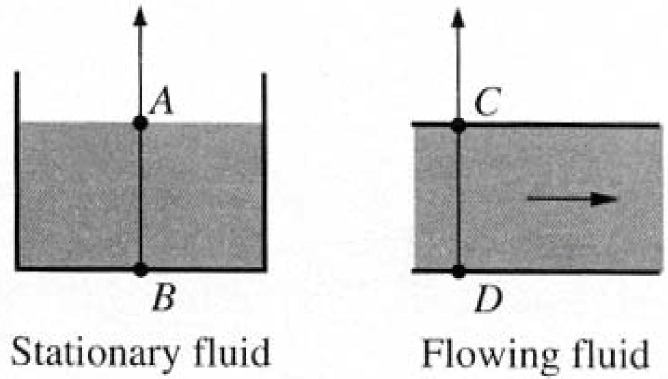


FIGURE 11-8

There is no fluid with zero viscosity, and thus the frictionless flow idealization is appropriate only when the viscous effects are relatively small.



$$P_A = P_C$$
$$P_B = P_D$$

FIGURE 11-13

The variation of pressure with elevation in steady, incompressible, frictionless flow along a straight line is the same as that in the stationary fluid (but this is not the case for a curved flow section).

5-6 The Bernoulli Equation (3)

$$\frac{P_1}{\rho} + \frac{V_1^{\,2}}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^{\,2}}{2} + gz_2$$

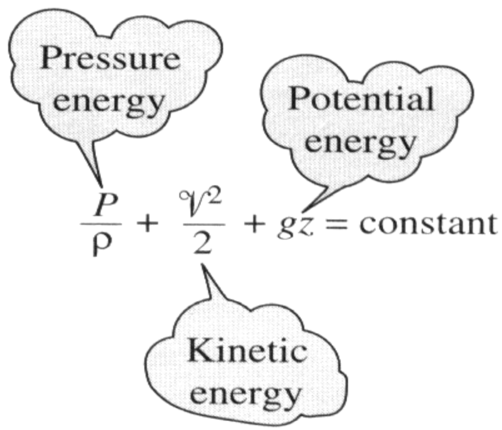


FIGURE 11-12

The Bernoulli equation states that the sum of the kinetic, potential, and pressure energies of a fluid particle is constant along a streamline during steady flow.

5-6 The Bernoulli Equation (4)

❑ Static,Dynamic, and Stagnation Pressure

$$P + \rho\frac{V^2}{2} + \rho g z = \text{constant} \quad \text{(kPa)}$$

$$P_{stag} = P + \rho\frac{V^2}{2} \qquad \text{(kPa)}$$

$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

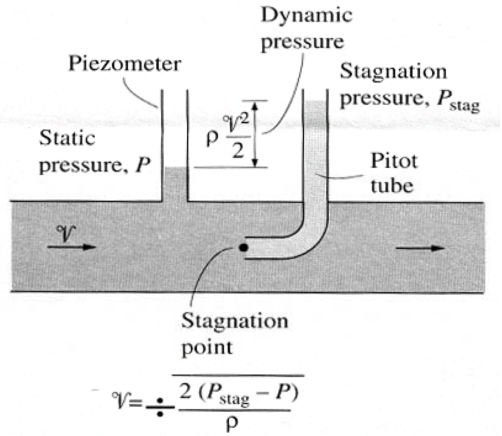


FIGURE 11-14

The static, dynamic, and stagnation pressures.

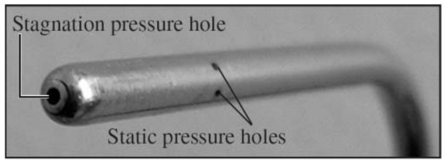


FIGURE 5–28

Close-up of a Pitot-static probe, showing the stagnation pressure hole and two of the five static circumferential pressure holes.

Photo by Po-Ya Abel Chuang. Used by permission.

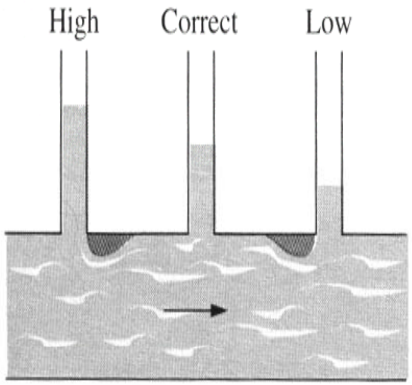
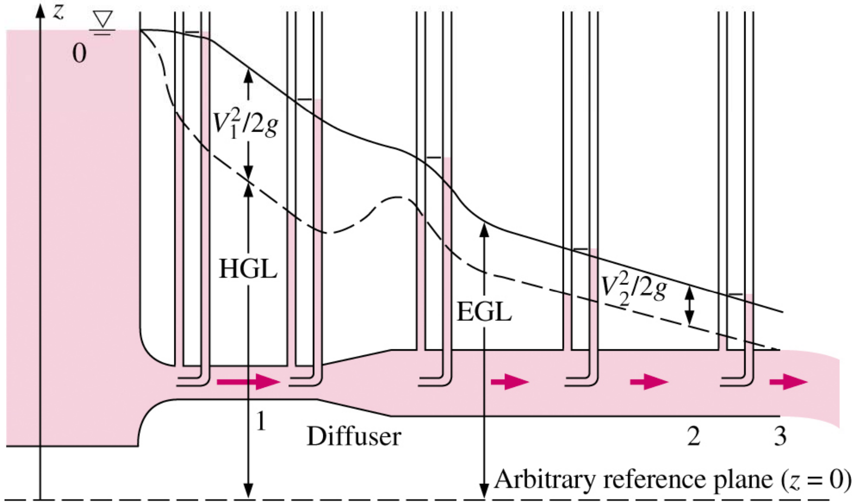
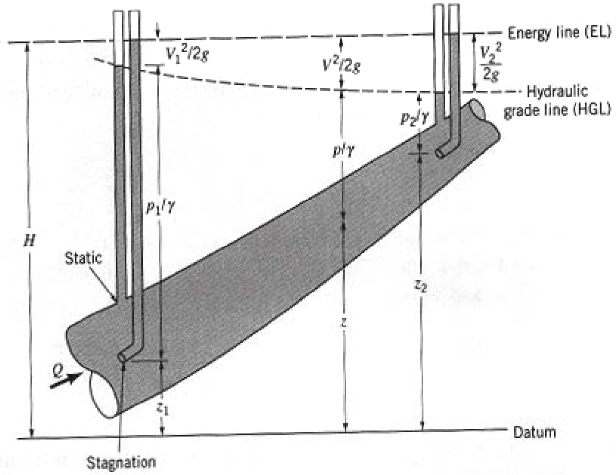


FIGURE 11-15

Careless placement of the pressure measurement device may result in erroneous reading of the static pressure.

5-6 The Bernoulli Equation (5)





■ **FIGURE 3.13** Representation of the energy line and the hydraulic grade line.

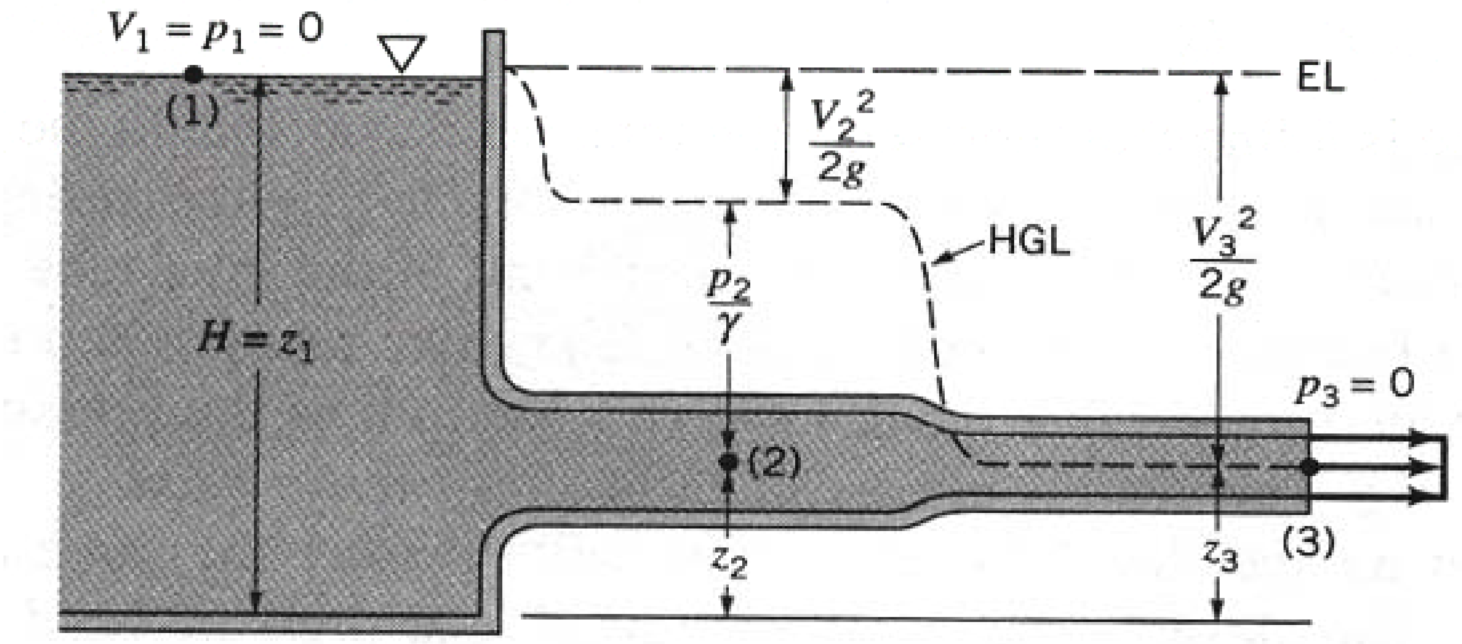
- It is often convenient to plot mechanical energy graphically using heights.
- Hydraulic Grade Line

$$HGL = \frac{P}{\rho g} + z$$

- Energy Grade Line (or total energy)

$$EGL = \frac{P}{\rho g} + \frac{V^2}{2g} + z$$

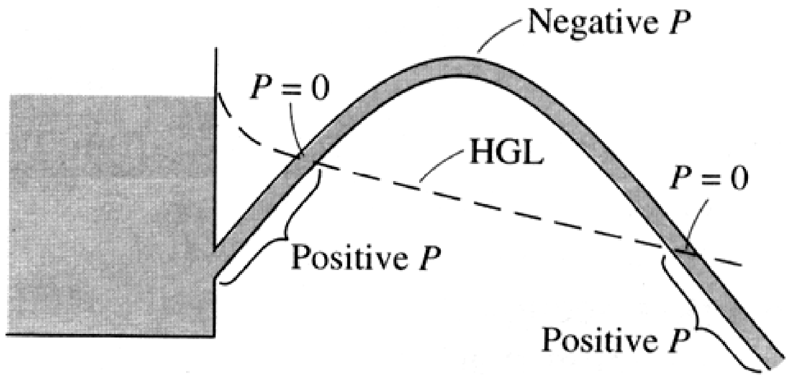
5-6 The Bernoulli Equation (6)

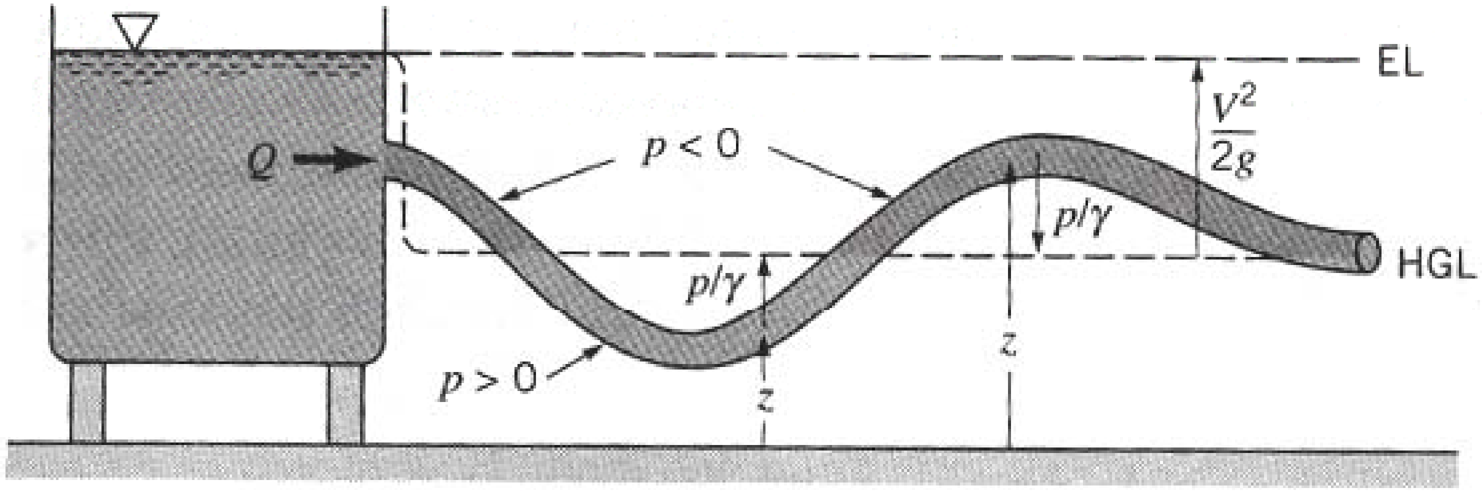


5-6 The Bernoulli Equation (7)

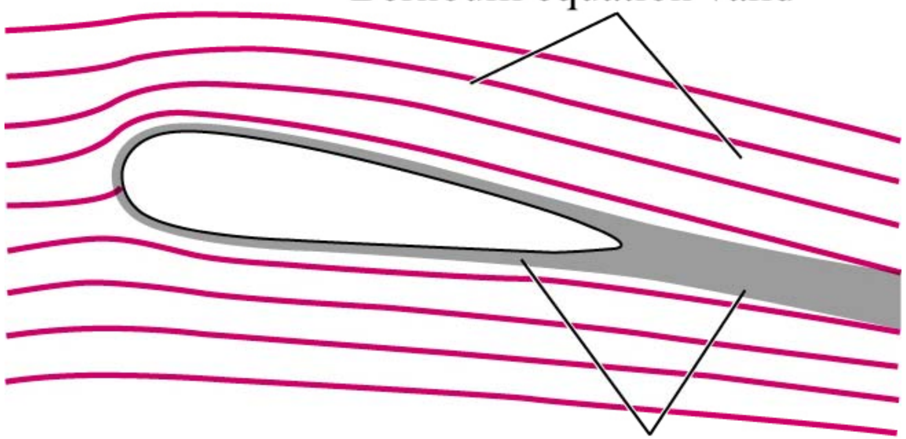
FIGURE 11-22

The pressure (gage) of a fluid is zero at locations where the HGL *intersects* the fluid, and the pressure is negative (vacuum) in a flow section that lies above the HGL.





5-6 The Bernoulli Equation (8)



- The **Bernoulli equation** is an *approximate relation between pressure, velocity*, and *elevation and is valid in regions of steady, incompressible flow where net frictional forces are negligible.*
- Equation is useful in flow regions outside of boundary layers and wakes.

5-6 The Bernoulli Equation (9)

- Limitations on the use of the Bernoulli Equation
  - Steady flow: $d/dt = 0$
  - Frictionless flow
  - No shaft work: $w_{pump}=w_{turbine}=0$
  - Incompressible flow: ρ = constant
  - No heat transfer: $q_{net,in}=0$
  - Applied along a streamline (except for irrotational flow)

5-6 The Bernoulli Equation (10)

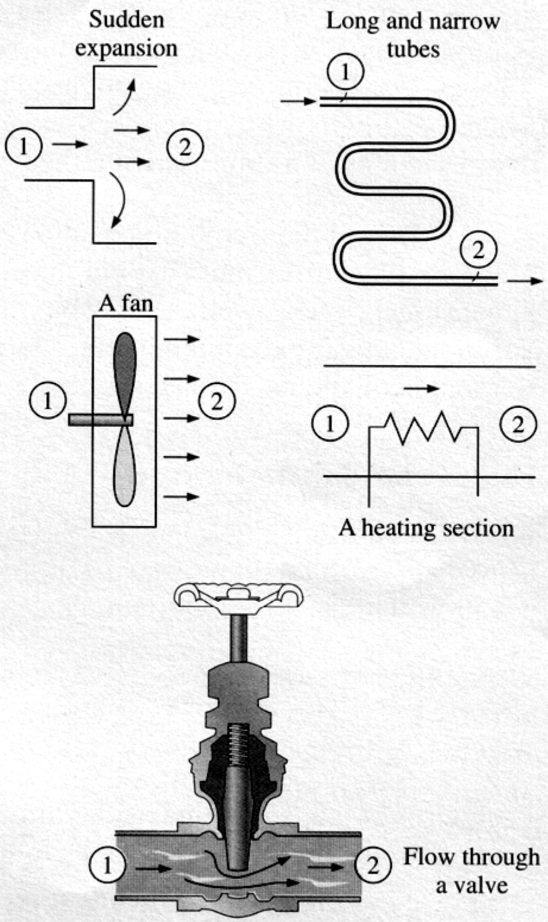
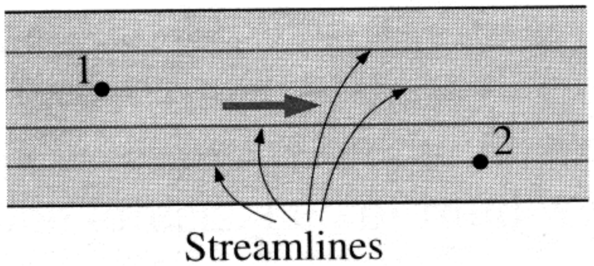


FIGURE 11-16

Frictional effects and components that disturb the streamlined structure of flow in a flow section can make the Bernoulli equation invalid.

5-6 The Bernoulli Equation (11)



$$\frac{P_1}{\rho} + \frac{\mathcal{V}_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{\mathcal{V}_2^2}{2} + gz_2$$

FIGURE 11-17

When the flow is irrotational, the Bernoulli equation becomes applicable between any two points along the flow (not necessarily on the same streamline).

5-6 The Bernoulli Equation (12)

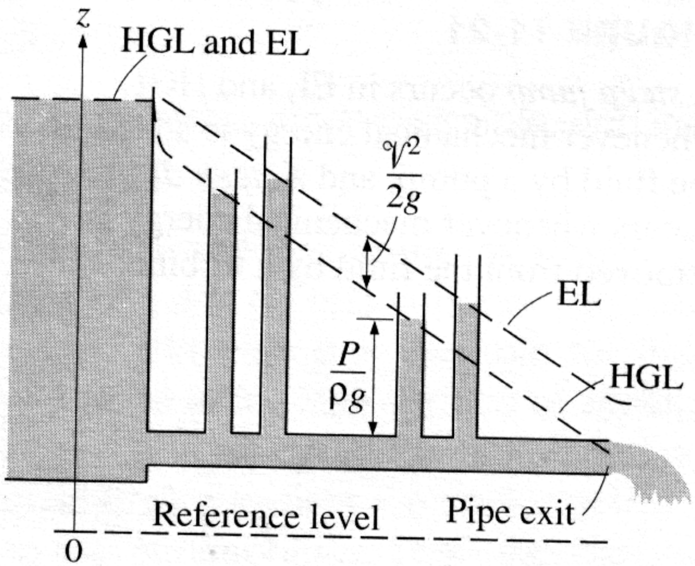


FIGURE 11-19

The hydraulic grade line (HGL) and the energy line (EL) for free discharge from a reservoir through a horizontal pipe (both HGL and EL decline due to pipe friction).

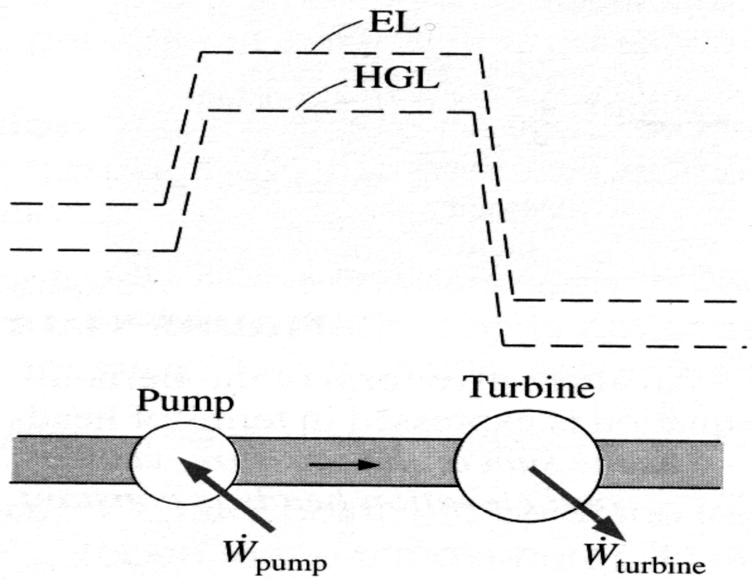


FIGURE 11-21

A *steep jump* occurs in EL and HGL whenever mechanical energy is added to the fluid by a pump, and a *steep drop* occurs whenever mechanical energy is removed from the fluid by a turbine.

5-6 The Bernoulli Equation (13)

$$e_{\text{mech,in}} = e_{\text{mech,out}} + e_{\text{mech,loss}}$$

$$\frac{P_1}{\rho g} + \frac{V_1^{\,2}}{2g} + z_1 + h_{\text{pump,u}} = \frac{P_2}{\rho g} + \frac{V_2^{\,2}}{2g} + z_2 + h_{\text{turbine}} + h_L$$

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FIGURE 11-30

When the frictional losses are negligible and there are no work devices involved, the steady-flow energy equation reduces to the Bernoulli equation.

Energy equation:

$$\frac{P_1}{\rho g} + \frac{V_1^{\,2}}{2g} + z_1 + \cancelto{0}{h_{\text{pump,u}}} = \frac{P_2}{\rho g} + \frac{V_2^{\,2}}{2g} + z_2 + \cancelto{0}{h_{\text{turbine}}} + \cancelto{0}{h_L}$$

Bernoulli equation:

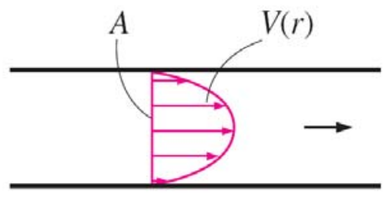
$$\frac{P_1}{\rho g} + \frac{V_1^{\,2}}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^{\,2}}{2g} + z_2$$

5-6 The Bernoulli Equation (14)

Kinetic Energy Correction Factor, α

$$\dot{m}\left(\frac{P_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right) + \dot{W}_{\text{pump}} = \dot{m}\left(\frac{P_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech, loss}} \quad \textbf{(5–76)}$$

$$\frac{P_1}{\rho g} + \alpha_1\frac{V_1^2}{2g} + z_1 + h_{\text{pump}, u} = \frac{P_2}{\rho g} + \alpha_2\frac{V_2^2}{2g} + z_2 + h_{\text{turbine}, e} + h_L \quad \textbf{(5–77)}$$



$$\dot{m} = \rho V_{\text{avg}} A, \qquad \rho = \text{constant}$$

$$\dot{KE}_{\text{act}} = \int \text{ke}\,\delta\dot{m} = \int_A \frac{1}{2} V^2(r)[\rho V(r)\, dA]$$

$$= \frac{1}{2}\rho \int_A V^3(r)\, dA$$

$$\dot{KE}_{\text{avg}} = \frac{1}{2}\dot{m}V_{\text{avg}}^2 = \frac{1}{2}\rho A V_{\text{avg}}^3$$

$$\alpha = \frac{\dot{KE}_{\text{act}}}{\dot{KE}_{\text{avg}}} = \frac{1}{A}\int_A \left(\frac{V(r)}{V_{\text{avg}}}\right)^3 dA$$

FIGURE 5–52

The determination of the *kinetic energy correction factor* using actual velocity distribution $V(r)$ and the average velocity V_{avg} at a cross section.

5-6 The Bernoulli Equation (15)

■ Derivation of the Bernoulli Equation

Applying Newton's second law (which is referred to as the *conservation of linear momentum* relation in fluid mechanics) in the *s*-direction on a particle moving along a streamline gives

$$\sum F_s = ma_s \tag{5–35}$$

5-6 The Bernoulli Equation (16)

$$P\,dA - (P + dP)\,dA - W\sin\theta = mV\,\frac{dV}{ds} \qquad \textbf{(5–36)}$$

$$-dP\,dA - \rho g\,dA\,ds\,\frac{dz}{ds} = \rho\,dA\,ds\,V\,\frac{dV}{ds} \qquad \textbf{(5–37)}$$

$$-dP - \rho g\,dz = \rho V\,dV \qquad \textbf{(5–38)}$$

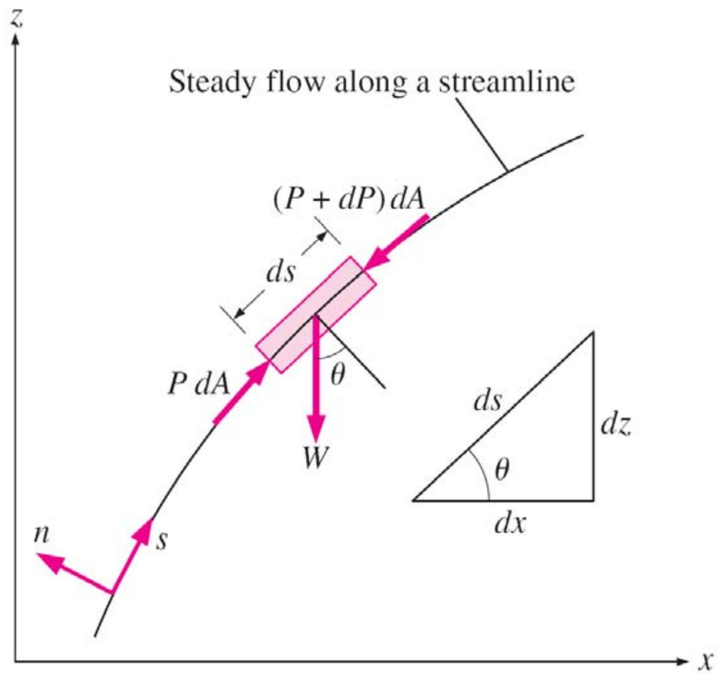


FIGURE 5–23
The forces acting on a fluid particle along a streamline.

5-6 The Bernoulli Equation (17)

$$\frac{dP}{\rho} + \tfrac{1}{2}\, d(V^2) + g\, dz = 0 \qquad \textbf{(5–39)}$$

Steady flow:

$$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)} \qquad \textbf{(5–40)}$$

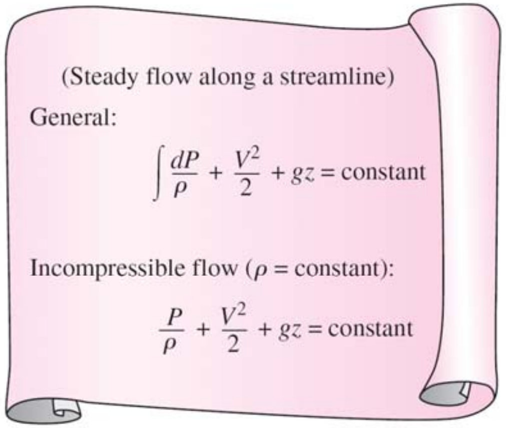


FIGURE 5–24
The Bernoulli equation is derived assuming incompressible flow, and thus it should not be used for flows with significant compressibility effects.

$$\frac{P}{\rho} + \int \frac{V^2}{R}\, dn + gz = \text{constant} \qquad \text{(across streamlines)} \qquad \textbf{(5–43)}$$

Steady, incompressible flow:

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant (along a streamline)} \qquad \textbf{(5–41)}$$

Steady, incompressible flow:

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 \qquad \textbf{(5–42)}$$