Chapter 4: Fluid Kinematics

4-1 Lagrangian and Eulerian Descriptions
4-2 Fundamentals of Flow Visualization
4-3 Kinematic Description
4-4 Reynolds Transport Theorem (RTT)

4-1 Lagrangian and Eulerian Descriptions (1)

- Lagrangian description of fluid flow tracks the position and velocity of individual particles.
- Based upon Newton's laws of motion.
- Difficult to use for practical flow analysis.
  - Fluids are composed of *billions* of molecules.
  - Interaction between molecules hard to describe/model.
- However, useful for specialized applications
  - Sprays, particles, bubble dynamics, rarefied gases.
  - Coupled Eulerian-Lagrangian methods.
- Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

4-1 Lagrangian and Eulerian Descriptions (2)

- Eulerian description of fluid flow: a **flow domain** or **control volume** is defined by which fluid flows in and out.
- We define **field variables** which are functions of space and time.
  - Pressure field, P=P(x,y,z,t)
  - Velocity field, $\vec{V} = \vec{V}(x, y, z, t)$

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

  - Acceleration field, $\vec{a} = \vec{a}(x, y, z, t)$

$$\vec{a} = a_x(x, y, z, t)\vec{i} + a_y(x, y, z, t)\vec{j} + a_z(x, y, z, t)\vec{k}$$

  - These (and other) field variables define the **flow field**.
- Well suited for formulation of initial boundary-value problems (PDE's).
- Named after Swiss mathematician Leonhard Euler (1707-1783).

4-1 Lagrangian and Eulerian Descriptions (3)

Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity. $$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$
- However, particle velocity at a point is the same as the fluid velocity, $$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$
- To take the time derivative of, chain rule must be used.

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

4-1 Lagrangian and Eulerian Descriptions (4)

- Since $\dfrac{dx_{particle}}{dt} = u, \dfrac{dy_{particle}}{dt} = v, \dfrac{dz_{particle}}{dt} = w$

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- In vector form, the acceleration can be written as

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

- First term is called the **local acceleration** and is nonzero only for unsteady flows.
- Second term is called the **advective acceleration** and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different.

4-1 Lagrangian and Eulerian Descriptions (5)

- The total derivative operator d/dt is call the **material derivative** and is often given special notation, D/Dt.

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

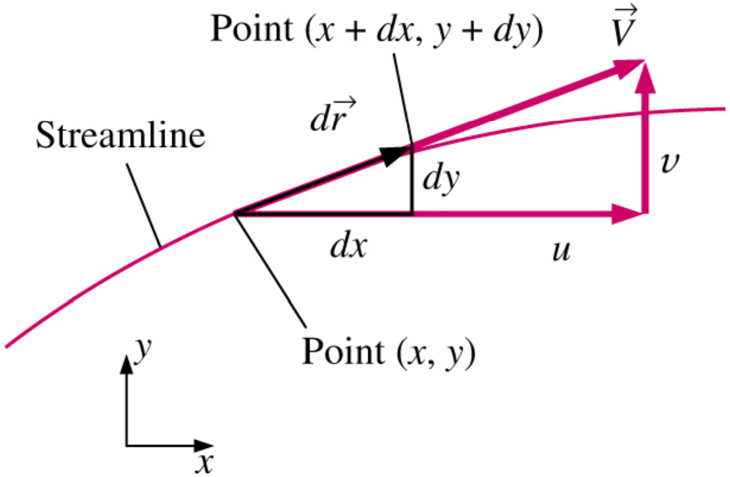
- Advective acceleration is nonlinear: source of many phenomenon and primary challenge in solving fluid flow problems.
- Provides ``transformation" between Lagrangian and Eulerian frames.
- Other names for the material derivative include: **total, particle, Lagrangian, Eulerian,** and **substantial** derivative.

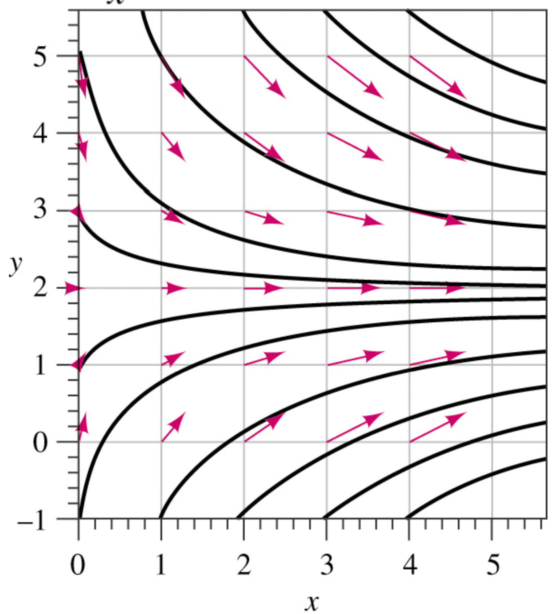
Steady: $\frac{\partial \vec{V}}{\partial t} = 0$

4-2 Fundamentals of Flow Visualization (1)

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive techniques
  - Surface flow techniques

4-2 Fundamentals of Flow Visualization (2)





- A **Streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an arc length

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- $d\vec{r}$ must be parallel to the local velocity vector

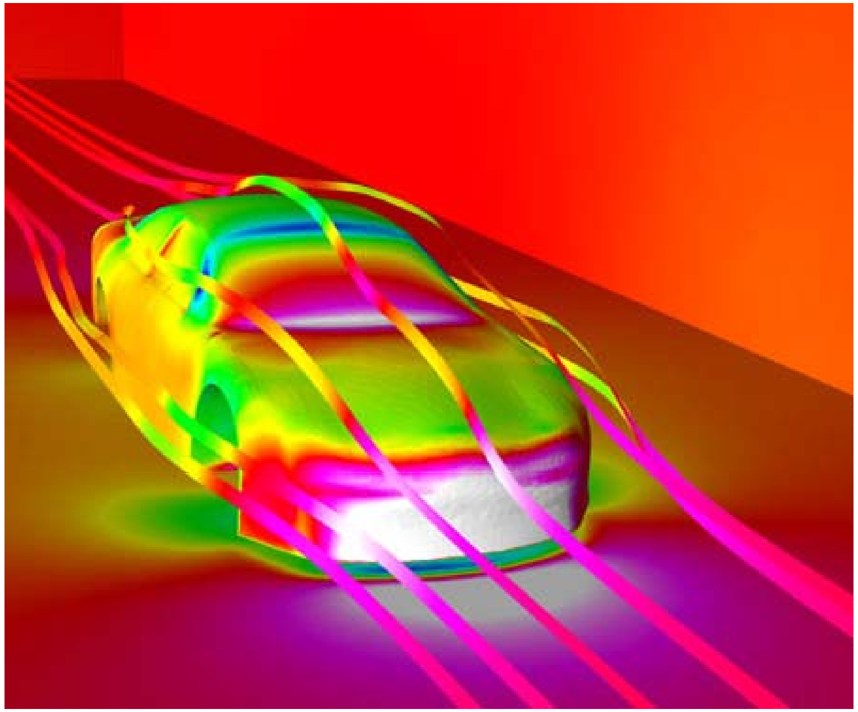
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Geometric arguments results in the equation for a streamline

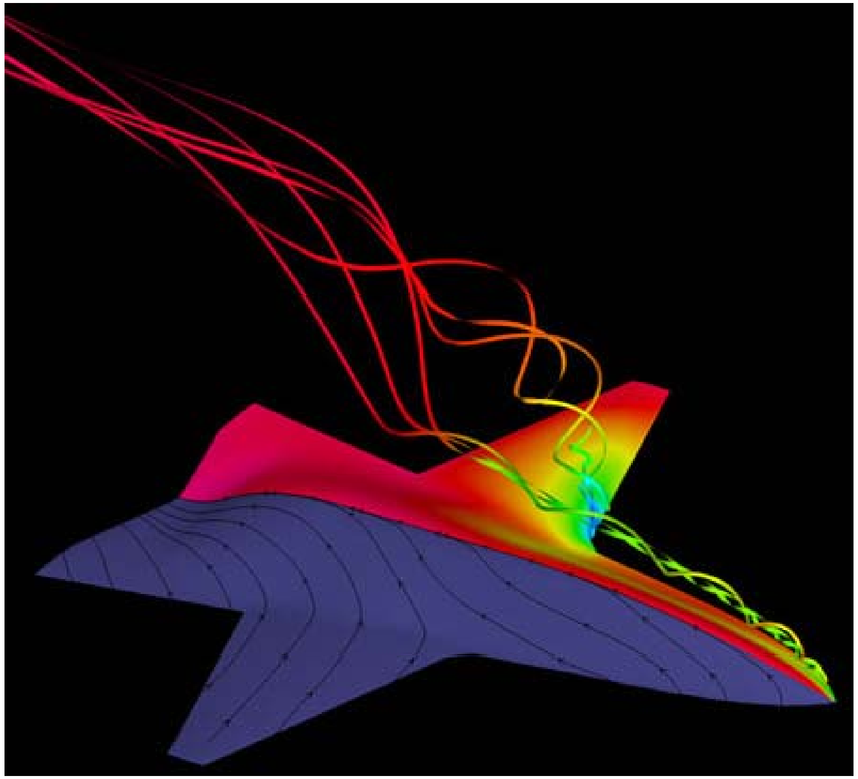
$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

4-2 Fundamentals of Flow Visualization (3)

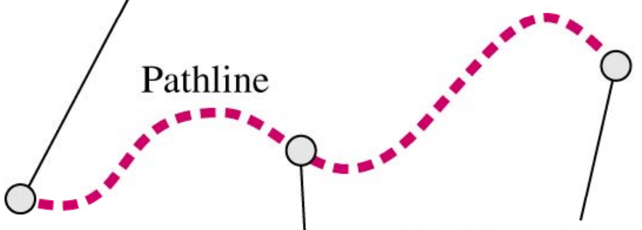
NASCAR surface pressure contours and streamlines



Airplane surface pressure contours, volume streamlines, and surface streamlines



4-2 Fundamentals of Flow Visualization (4)



$$\left.\frac{dx}{dt}\right)_{particle} = u(x, y, z, t)$$

$$\left.\frac{dy}{dt}\right)_{particle} = \upsilon(x, y, z, t)$$

$$\left.\frac{dz}{dt}\right)_{particle} = w(x, y, z, t)$$

- A **Pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

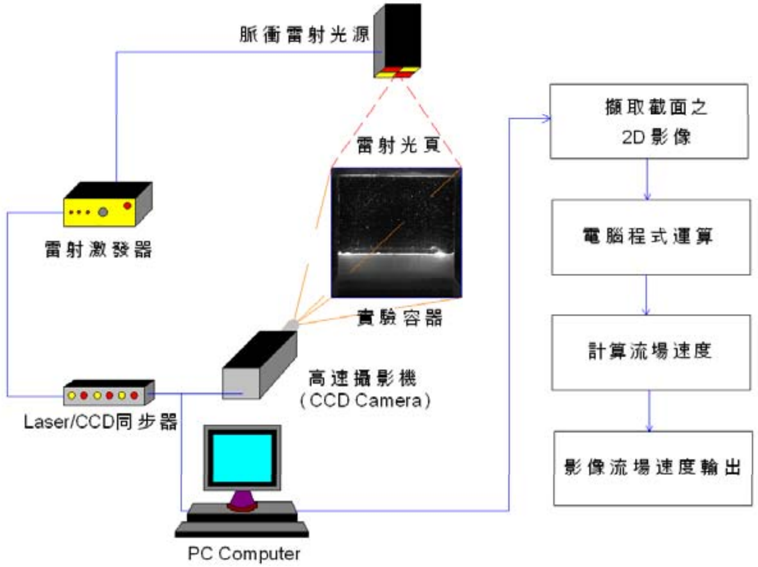
$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

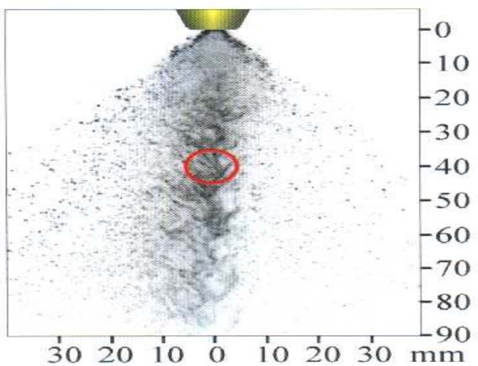
- Particle location at time t:

$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

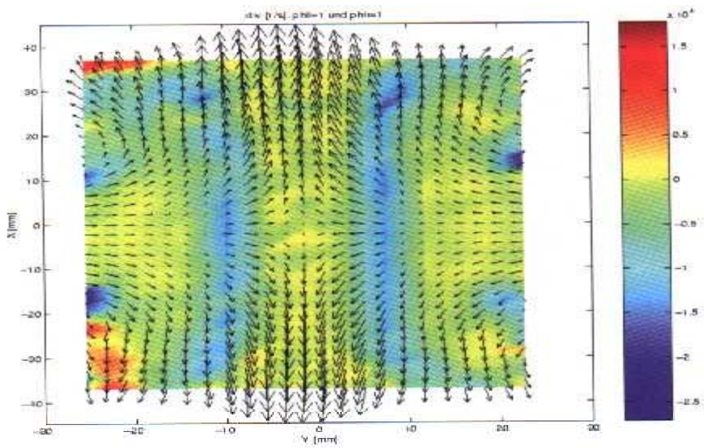
- Particle Image Velocimetry (PIV) is a modern experimental technique to measure velocity field over a plane in the flow field.

4-2 Fundamentals of Flow Visualization (5)

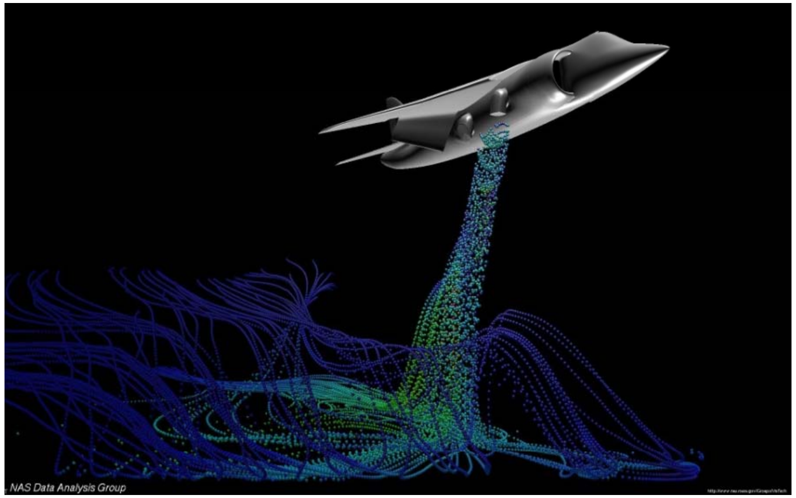


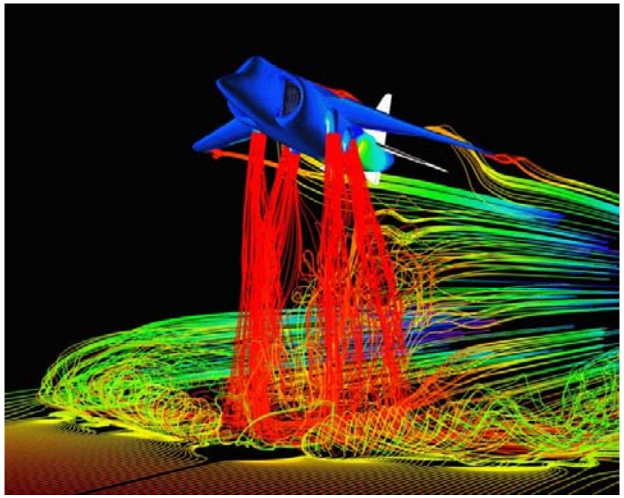


LIF (Mie) signal from a fuel injector



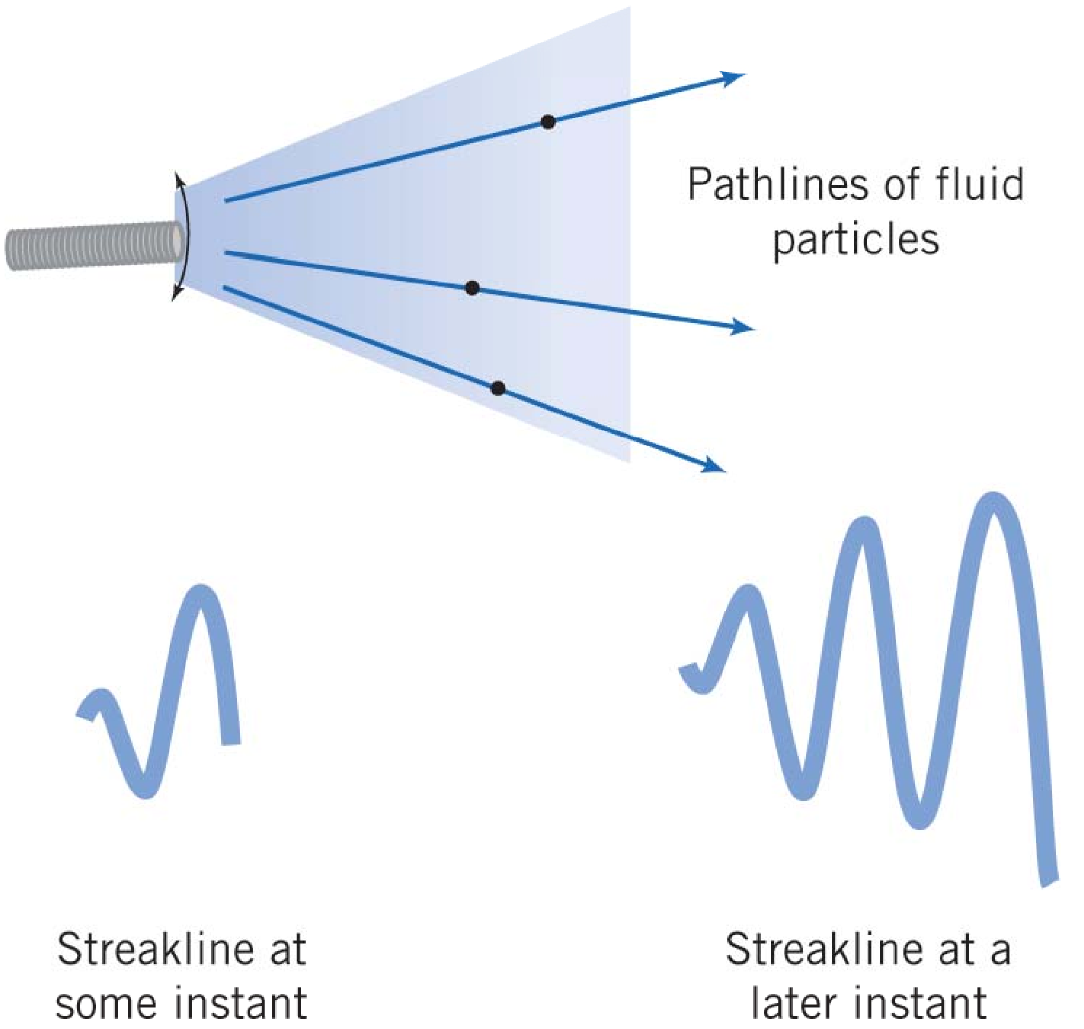
4-2 Fundamentals of Flow Visualization (6)





- A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.

4-2 Fundamentals of Flow Visualization (7)



4-2 Fundamentals of Flow Visualization (8)

Comparison

- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline: instantaneous snapshot of a time-integrated flow pattern.
  - Pathline: time-exposed flow path of an individual particle.

4-2 Fundamentals of Flow Visualization (9)

- **EXAMPLE2.1** Streamlines and Pathlines in Two-Dimensional Flow

  A velocity field is given by ; the units of velocity are m/s; *x* and *y* are given in meters; .

(a) Obtain an equation for the streamlines in the *xy* plane.

(b) Plot the Streamlines passing through the point

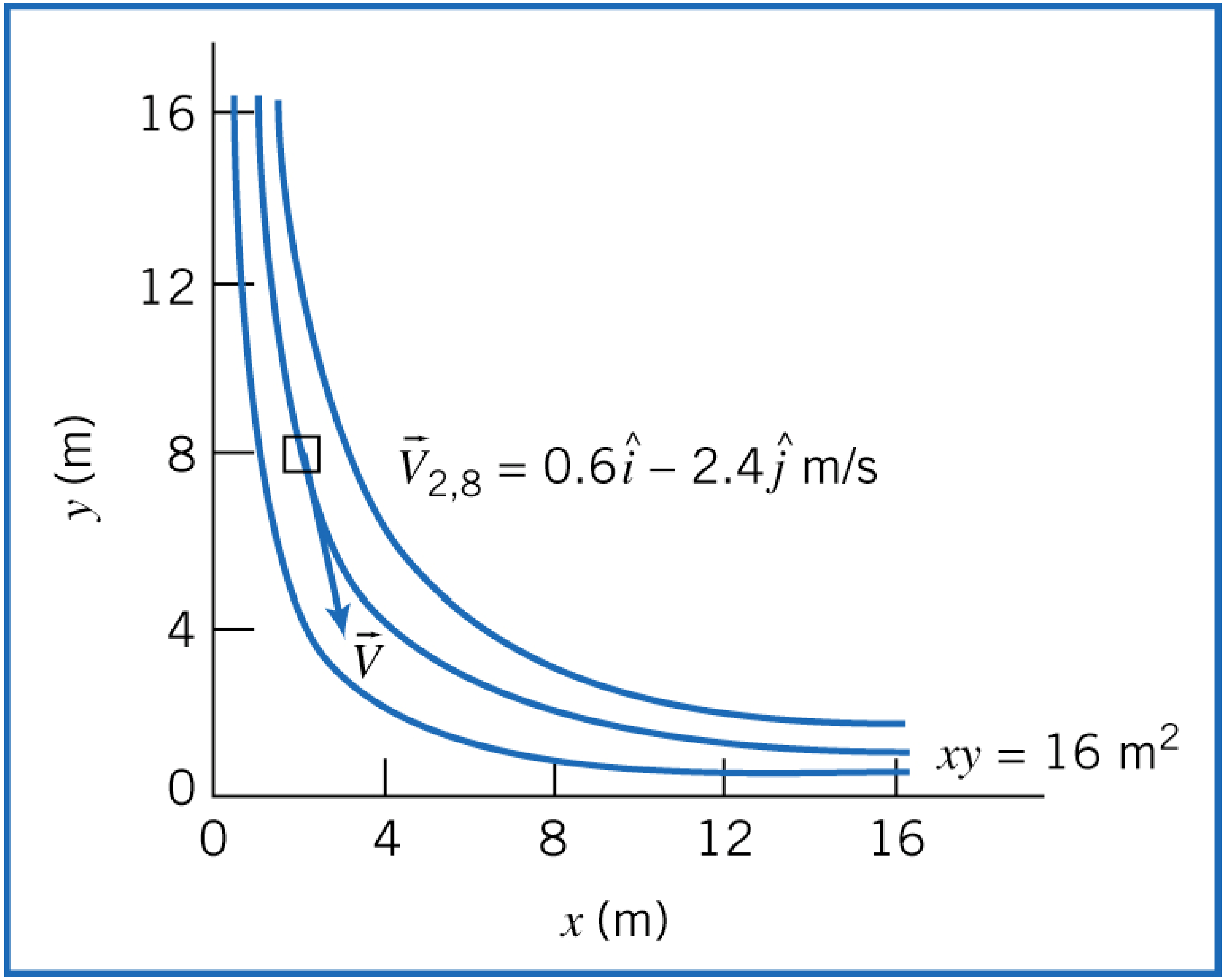
(c) Determine the velocity of particle at the point (2, 8).

(d) If the particle passing through the point (is marked at time $t = 0$, determine the location of the particle at time $t = 6$ s .

(e) What is the velocity of this particle at time $t = 6$ s ?

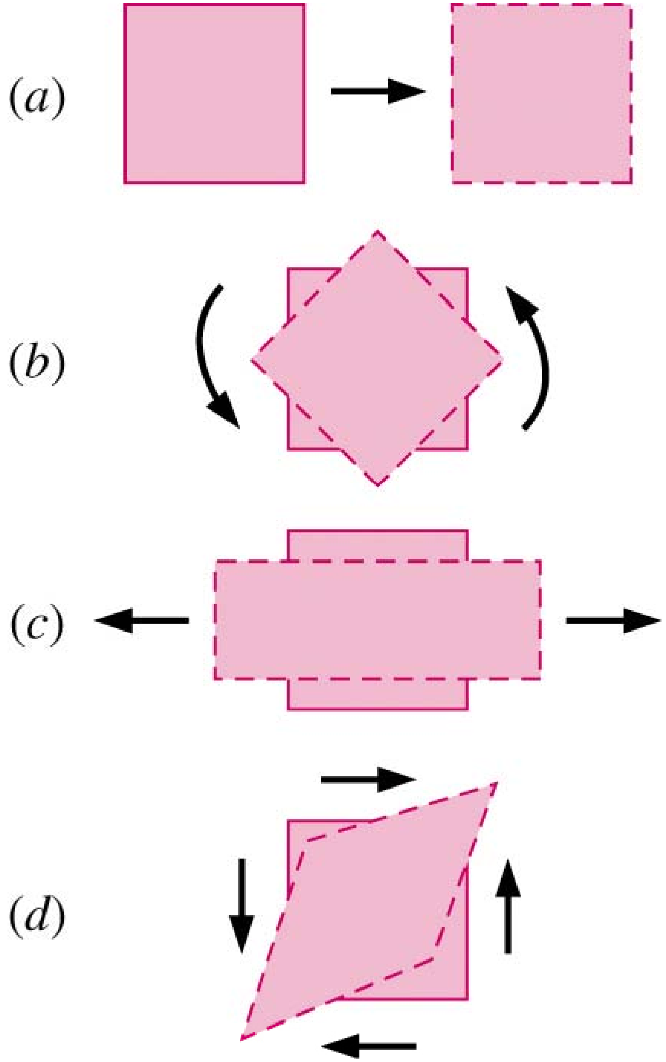
(f) Show that the equation of the particle path (the pathline) is the same as the equation of the Streamline.

4-2 Fundamentals of Flow Visualization (10)



4-15

4-3 Kinematic Description (1)



- In fluid mechanics, an element may undergo four fundamental types of motion.
  a) Translation
  b) Rotation
  c) Linear strain
  d) Shear strain
- Because fluids are in constant motion, motion and deformation is best described in terms of rates
  a) velocity: rate of translation
  b) angular velocity: rate of rotation
  c) linear strain rate: rate of linear strain
  d) shear strain rate: rate of shear strain

4-3 Kinematic Description (2)

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described as the velocity vector. In Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- **Rate of rotation** at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

4-3 Kinematic Description (3)

- **Linear Strain Rate** is defined as the rate of increase in length per unit length.
- In Cartesian coordinates

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.

4-3 Kinematic Description (4)

- **Shear Strain Rate** at a point is defined as *half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point.*
- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

4-3 Kinematic Description (5)

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

4-3 Kinematic Description (6)

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion
- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.
  - Feature extraction and flow visualization in CFD simulations.

4-3 Kinematic Description (7)

- The **vorticity vector** is defined as the curl of the velocity vector $\vec{\zeta} = \vec{\nabla} \times \vec{V}$
- Vorticity is equal to twice the angular velocity of a fluid particle. $\vec{\zeta} = 2\vec{\omega}$
  Cartesian coordinates
  $$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$
  Cylindrical coordinates
  $$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$
- In regions where z = 0, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

4-3 Kinematic Description (8)

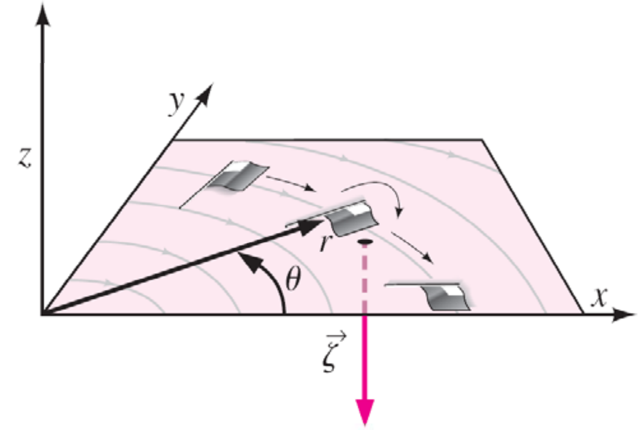
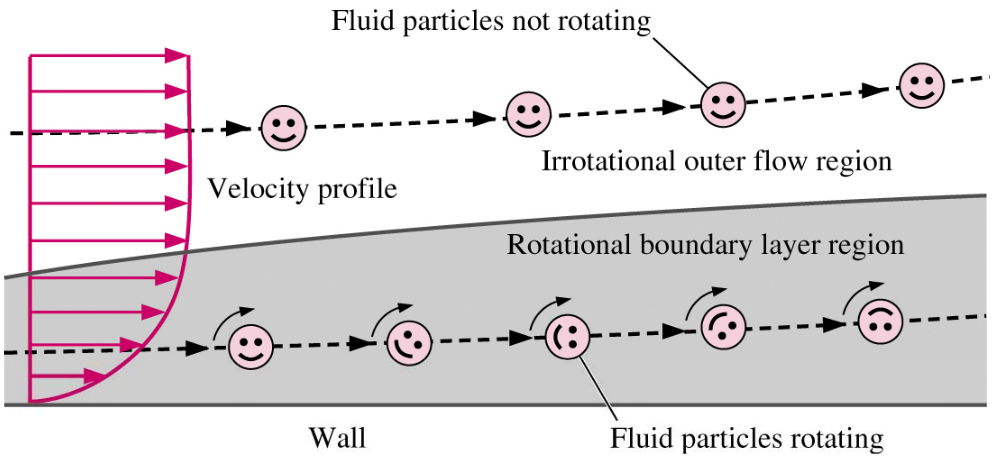


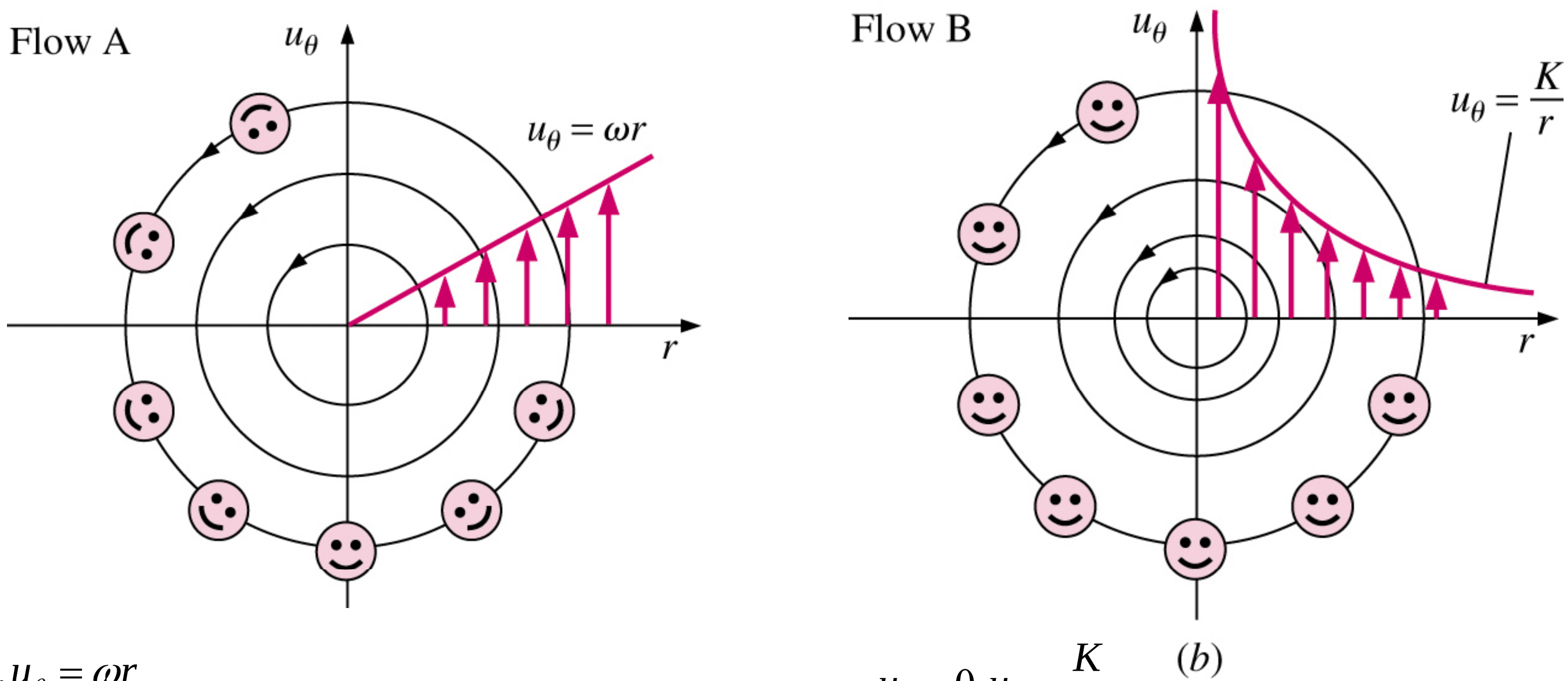
FIGURE 4–48

For a two-dimensional flow in the $r\theta$-plane, the vorticity vector always points in the z (or $-z$) direction. In this illustration, the flag-shaped fluid particle rotates in the clockwise direction as it moves in the $r\theta$-plane; its vorticity points in the $-z$-direction as shown.



4-3 Kinematic Description (9)

Special case: consider two flows with circular streamlines



$$u_r = 0, u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

4-4 Reynolds Transport Theorem (RTT) (1)

- A **system** is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A **control volume** is a region in space chosen for study. Mass can cross a control surface.
- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).

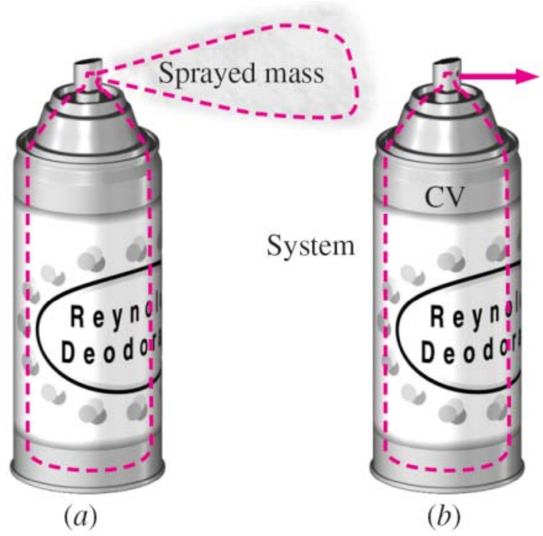


FIGURE 4–52
Two methods of analyzing the spraying of deodorant from a spray can: (*a*) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed. (*b*) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

4-4 Reynolds Transport Theorem (RTT) (2)

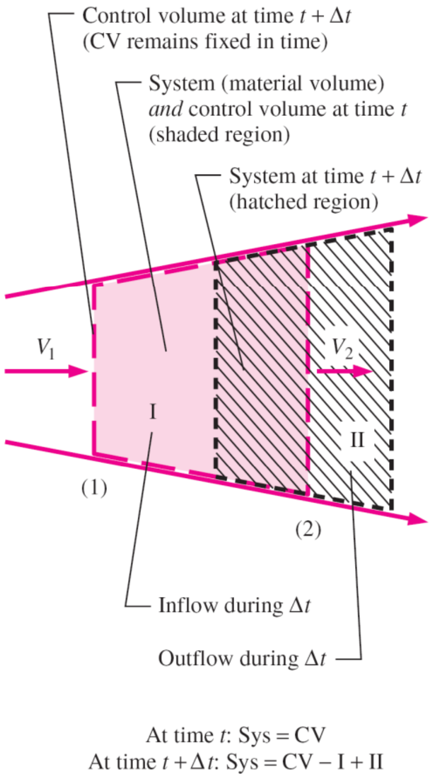


FIGURE 4–54

A moving *system* (hatched region) and a fixed *control volume* (shaded region) in a diverging portion of a flow field at times t and $t + \Delta t$. The upper and lower bounds are streamlines of the flow.

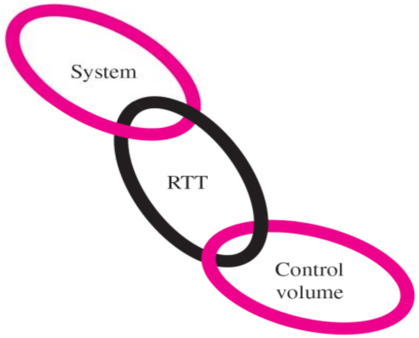


FIGURE 4–53

The *Reynolds transport theorem* (RTT) provides a link between the system approach and the control volume approach.

Let B represent any **extensive property** (such as mass, energy, or momentum), and let $b=B/m$ represent the corresponding **intensive property.** Noting that extensive properties are additive, the extensive property B of the system at times t and t t can be expressed as

$$B_{\text{sys},\,t} = B_{\text{CV},\,t} \qquad \text{(the system and CV concide at time } t\text{)}$$

$$B_{\text{sys},\,t+\Delta t} = B_{\text{CV},\,t+\Delta t} - B_{\text{I},\,t+\Delta t} + B_{\text{II},\,t+\Delta t}$$

4-4 Reynolds Transport Theorem (RTT) (3)

$$\frac{B_{\text{sys},\,t+\Delta t} - B_{\text{sys},\,t}}{\Delta t} = \frac{B_{\text{CV},\,t+\Delta t} - B_{\text{CV},\,t}}{\Delta t} - \frac{B_{\text{I},\,t+\Delta t}}{\Delta t} + \frac{B_{\text{II},\,t+\Delta t}}{\Delta t}$$

Taking the limit as $\Delta t \to 0$, and using the definition of derivative, we get

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - \dot{B}_{\text{in}} + \dot{B}_{\text{out}} \qquad \textbf{(4–38)}$$

or

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{CV}}}{dt} - b_1\rho_1 V_1 A_1 + b_2\rho_2 V_2 A_2$$

since

$$B_{\text{I},\,t+\Delta t} = b_1 m_{\text{I},\,t+\Delta t} = b_1\rho_1 V_{\text{I},\,t+\Delta t} = b_1\rho_1 V_1\,\Delta t\,A_1$$

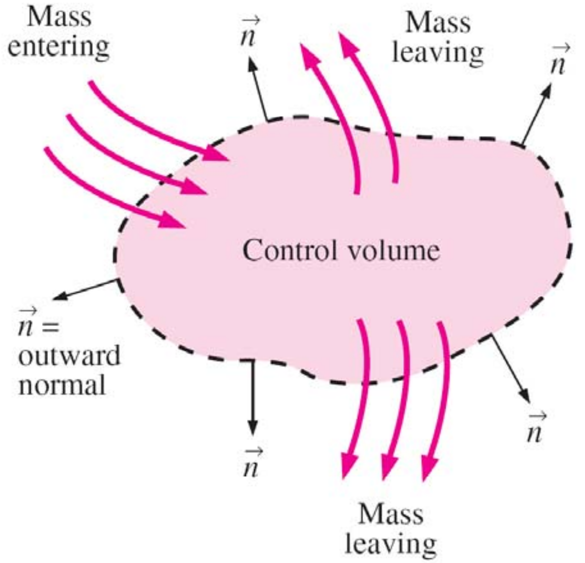
$$B_{\text{II},\,t+\Delta t} = b_2 m_{\text{II},\,t+\Delta t} = b_2\rho_2 V_{\text{II},\,t+\Delta t} = b_2\rho_2 V_2\,\Delta t\,A_2$$

and

$$\dot{B}_{\text{in}} = \dot{B}_{\text{I}} = \lim_{\Delta t\to 0}\frac{B_{\text{I},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{b_1\rho_1 V_1\,\Delta t\,A_1}{\Delta t} = b_1\rho_1 V_1 A_1$$

$$\dot{B}_{\text{out}} = \dot{B}_{\text{II}} = \lim_{\Delta t\to 0}\frac{B_{\text{II},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t\to 0}\frac{b_2\rho_2 V_2\,\Delta t\,A_2}{\Delta t} = b_2\rho_2 V_2 A_2$$

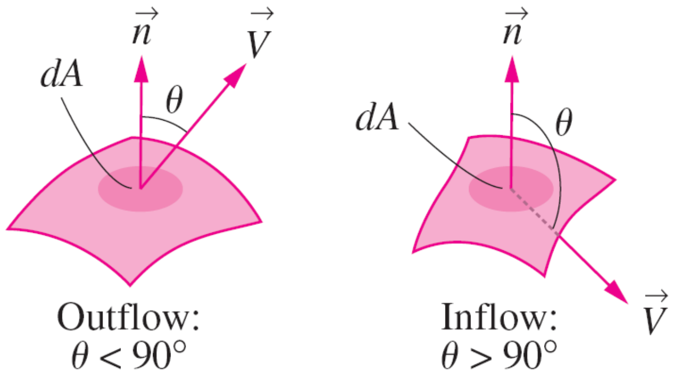
4-4 Reynolds Transport Theorem (RTT) (4)



$$\dot{B}_{\text{net}} = \dot{B}_{\text{out}} - \dot{B}_{\text{in}} = \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA$$

FIGURE 4–55

The integral of $b\rho\vec{V} \cdot \vec{n}\, dA$ over the control surface gives the net amount of the property B flowing out of the control volume (into the control volume if it is negative) per unit time.



$\vec{V} \cdot \vec{n} = |\vec{V}||\vec{n}| \cos\theta = V\cos\theta$

If $\theta < 90°$, then $\cos\theta > 0$ (outflow).

If $\theta > 90°$, then $\cos\theta < 0$ (inflow).

If $\theta = 90°$, then $\cos\theta = 0$ (no flow).

FIGURE 4–56

Outflow and inflow of mass across the differential area of a control surface.

4-4 Reynolds Transport Theorem (RTT) (5)

$$B_{CV} = \int_{CV} \rho b \, dV \qquad (4\text{–}40)$$

RTT, fixed CV:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA \qquad (4\text{–}41)$$

Alternate RTT, fixed CV:

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA \qquad (4\text{–}42)$$

4-4 Reynolds Transport Theorem (RTT) (6)

- Interpretation of the RTT:
  - Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)
  - Term 1: the time rate of change of B of the control volume
  - Term 2: the net flux of B out of the control volume by mass crossing the control surface

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\,d\forall + \int_{CS} \rho b \vec{V} \cdot \vec{n}\,dA$$

4-4 Reynolds Transport Theorem (RTT) (7)

- **Special Case 1: Steady Flow**

$$Steady\ \ flow: \qquad \frac{dB_{sys}}{dt} = \int_{\mathrm{CS}} \rho b(\vec{V} \cdot \hat{n}) dA$$

- **Special Case 2: One-Dimensional Flow**

$$One\text{-}dimensional\ \ flow:$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b\, dV + \sum_{\mathrm{out}} \underbrace{\rho_e b_e V_e A_e}_{\text{for each exit}} - \sum_{\mathrm{in}} \underbrace{\rho_i b_i V_i A_i}_{\text{for each exit}}$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b\, dV + \sum_{\mathrm{out}} \dot{m}_e\, b_e - \sum_{\mathrm{in}} \dot{m}_i\, b_i$$

4-4 Reynolds Transport Theorem (RTT) (8)

- Material derivative (differential analysis):

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) b$$

- General RTT, nonfixed CV (integral analysis):

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}\left(\rho b\right) d\mathcal{V} + \int_{CS} \rho b \vec{V} \cdot \vec{n} dA$$

	Mass	Momentum	Energy	Angular momentum
B, Extensive properties	m	$m\vec{V}$	E	$\vec{H}$
b, Intensive properties	1	$\vec{V}$	e	$\left(\vec{r} \times \vec{V}\right)$

- In Chaps 5 and 6, we will apply RTT to conservation of mass, energy, linear momentum, and angular momentum.

4-4 Reynolds Transport Theorem (RTT) (9)

Equation 4–41 was derived for a *fixed* control volume. However, many practical systems such as turbine and propeller blades involve nonfixed control volumes. Fortunately, Eq. 4–41 is also valid for *moving* and/or *deforming* control volumes provided that the absolute fluid velocity $\vec{V}$ in the last term be replaced by the **relative velocity** $\vec{V}_r$,

Relative velocity:

$$\vec{V}_r = \vec{V} - \vec{V}_{CS} \tag{4–43}$$

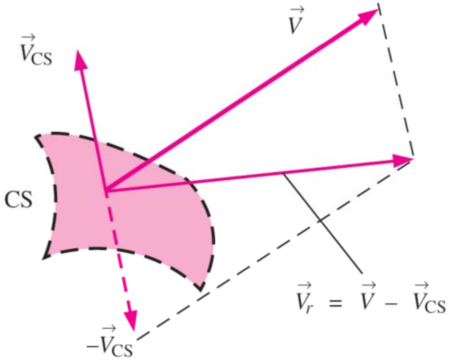


FIGURE 4–57
Relative velocity crossing a control surface is found by vector addition of the absolute velocity of the fluid and the negative of the local velocity of the control surface.

RTT, nonfixed CV:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA \tag{4–44}$$

4-4 Reynolds Transport Theorem (RTT) (10)

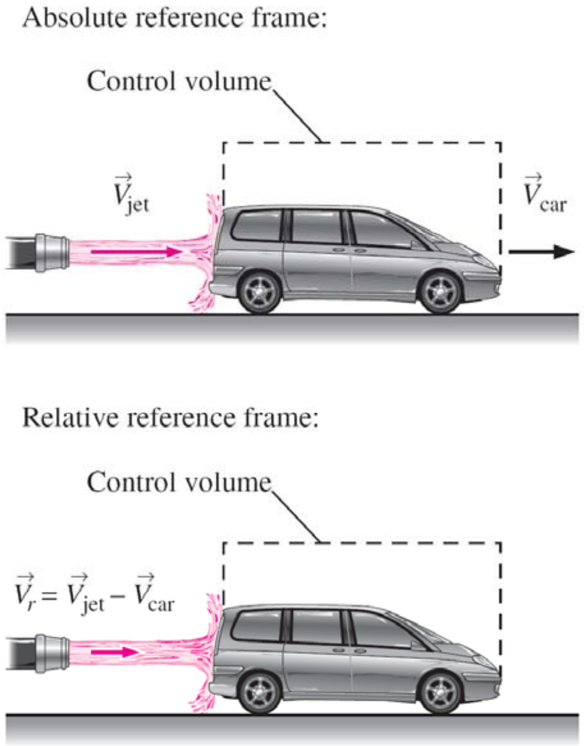


FIGURE 4–58
Reynolds transport theorem applied to a control volume moving at constant velocity.

4-4 Reynolds Transport Theorem (RTT) (11)

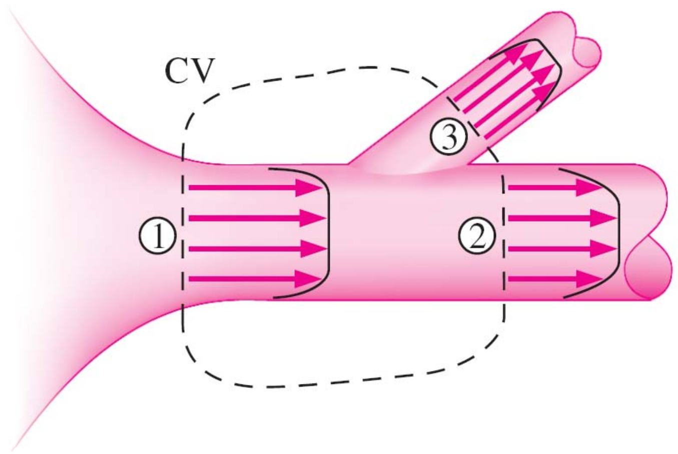


FIGURE 4–59

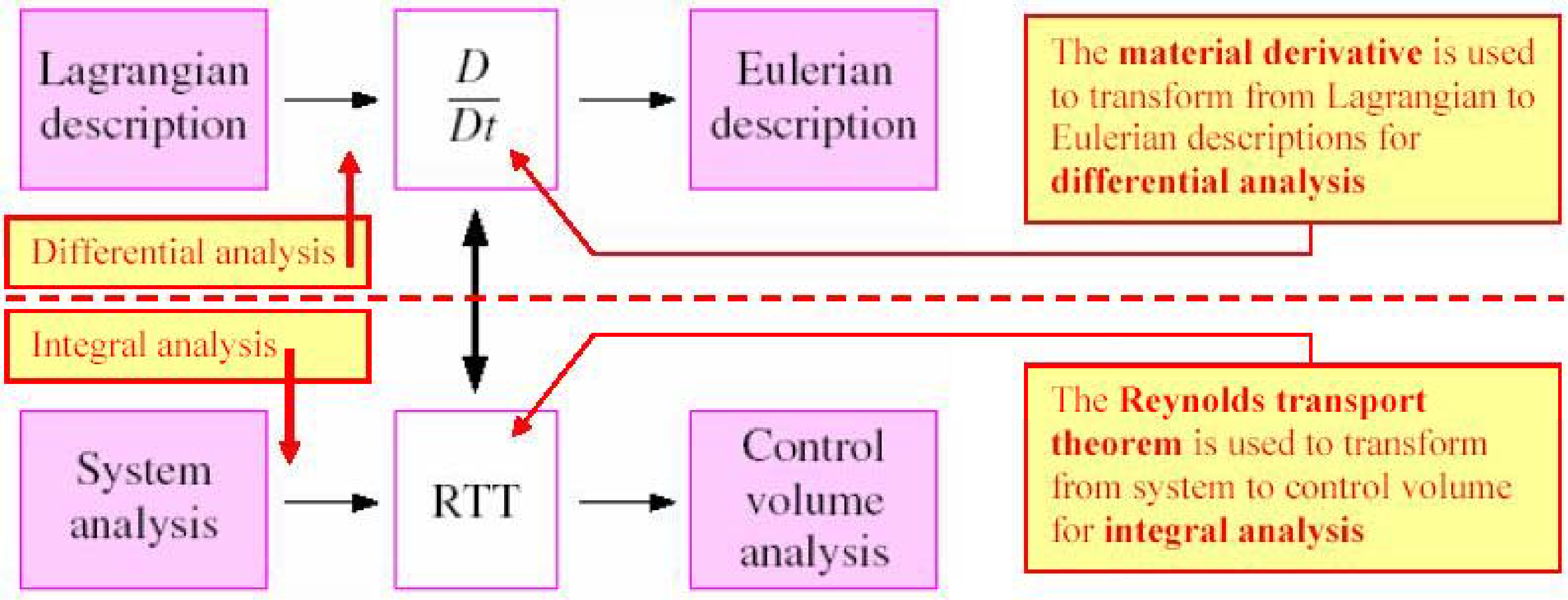
An example control volume in which there is one well-defined inlet (1) and two well-defined outlets (2 and 3). In such cases, the control surface integral in the RTT can be more conveniently written in terms of the average values of fluid properties crossing each inlet and outlet.

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \sum_{out} \underbrace{\dot{m}_r b_{avg}}_{\text{for each outlet}} - \sum_{in} \underbrace{\dot{m}_r b_{avg}}_{\text{for each inlet}} \qquad (4\text{–}47)$$

RTT, steady flow:

$$\frac{dB_{sys}}{dt} = \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA \qquad (4\text{–}46)$$

4-4 Reynolds Transport Theorem (RTT) (12)



There is a direct analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields).