Fluid Mechanics HW#2 Solution

4-15

Solution We are to calculate the material acceleration for a given velocity field.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *x*-*y* plane.

Analysis The velocity field is

$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

Material acceleration components:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + (U_0 + bx)b + (-by)0 + 0$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 + (U_0 + bx)0 + (-by)(-b) + 0 \qquad (2)$$

where the unsteady terms are zero since this is a steady flow, and the terms with w are zero since the flow is two-dimensional. Eq. 2 simplifies to

Material acceleration components: $a_x = b(U_0 + bx) \qquad a_y = b^2 y$ (3)

In terms of a vector,

Material acceleration vector: $\vec{a} = b(U_0 + bx)\vec{i} + b^2 y\vec{j}$ (4)

4-23

Solution For a given velocity field we are to generate an equation for the streamlines.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *x*-*y* plane.

Analysis The steady, two-dimensional velocity field of Problem 4-15 is

Velocity field:
$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

For two-dimensional flow in the *x*-*y* plane, streamlines are given by

Streamlines in the x-y plane:
$$\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \qquad (2)$$

We substitute the u and v components of Eq. 1 into Eq. 2 and rearrange to get

$$\frac{dy}{dx} = \frac{-by}{U_0 + bx}$$

We solve the above differential equation by separation of variables:

$$-\int \frac{dy}{by} = \int \frac{dx}{U_0 + bx}$$

Integration yields

$$-\frac{1}{b}\ln(by) = \frac{1}{b}\ln(U_0 + bx) + \frac{1}{b}\ln C_1 \qquad (3)$$

where we have set the constant of integration as the natural logarithm of some constant C_1, with a constant in front in order to simplify the algebra (notice that the factor of $1/b$ can be removed from each term in Eq. 3). When we recall that $\ln(ab) = \ln a + \ln b$, and that $-\ln a = \ln(1/a)$, Eq. 3 simplifies to

Equation for streamlines:
$$\boxed{y = \frac{C}{(U_0 + bx)}} \qquad (4)$$

The new constant C is related to C_1, and is introduced for simplicity.

4-43

Solution For a given velocity field we are to generate an equation for the x location of a fluid particle along the x-axis as a function of time.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the x-y plane.

Analysis The velocity field is

Velocity field:
$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

We start with the definition of u following a fluid particle,

x-component of velocity of a fluid particle:
$$\frac{dx_{\text{particle}}}{dt} = u = U_0 + bx_{\text{particle}} \qquad (2)$$

where we have substituted u from Eq. 1. We rearrange and separate variables, dropping the "particle" subscript for convenience,

$$\frac{dx}{U_0 + bx} = dt \qquad (3)$$

Integration yields

$$\frac{1}{b}\ln(U_0 + bx) = t - \frac{1}{b}\ln C_1 \qquad (4)$$

where we have set the constant of integration as the natural logarithm of some constant C_1, with a constant in front in order to simplify the algebra. When we recall that $\ln(ab) = \ln a + \ln b$, Eq. 4 simplifies to

$$\ln\left(C_1\left(U_0 + bx\right)\right) = t$$

from which

$$U_0 + bx = C_2 e^{bt} \qquad (5)$$

where C_2 is a new constant defined for convenience. We now plug in the known initial condition that at $t = 0$, $x = x_A$ to find constant C_2 in Eq. 5. After some algebra,

Fluid particle's x location at time t:
$$\boxed{x = x_{A'} = \frac{1}{b}\left[\left(U_0 + bx_A\right)e^{bt} - U_0\right]} \qquad (6)$$

4-58

Solution For the given velocity field we are to calculate the two-dimensional linear strain rates from fundamental principles and compare with the given equation.

Assumptions **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

Analysis First, for convenience, we number the equations in the problem statement:

Velocity field: $$\vec{V} = (u, v) = (a + by)\vec{i} + 0\vec{j} \qquad (1)$$

Lower left corner at t + dt: $$\left(x + (a + by)dt,\, y\right) \qquad (2)$$

Linear strain rate in Cartesian coordinates: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad (3)$$

(*a*) The lower right corner of the fluid particle moves the same amount as the lower left corner since *u* does not depend on *y* position. Thus,

Lower right corner at t + dt: $$\left(x + dx + (a + by)dt,\, y\right) \qquad (4)$$

Similarly, the top two corners of the fluid particle move to the right at speed $a + b(y+dy)dt$. Thus,

Upper left corner at t + dt: $$\left(x + \left(a + b\left(y + dy\right)\right)dt,\, y + dy\right) \qquad (5)$$

and

Upper right corner at t + dt: $$\left(x + dx + \left(a + b\left(y + dy\right)\right)dt,\, y + dy\right) \qquad (6)$$

(*b*) From the fundamental definition of linear strain rate in the *x*-direction, we consider the lower edge of the fluid particle. Its rate of increase in length divided by its original length is found by using Eqs. 2 and 4,

ε_{xx}: $$\varepsilon_{xx} = \frac{1}{dt}\left[\frac{\overbrace{x + dx + (a + by)dt - \left(x + (a + by)dt\right)}^{\text{Length of lower edge at } t+dt} - \overbrace{dx}^{\text{Length of lower edge at } t}}{dx}\right] = 0 \qquad (6)$$

We get the same result by considering the *upper* edge of the fluid particle. Similarly, using the left edge of the fluid particle and Eqs. 2 and 5 we get

ε_{yy}: $$\varepsilon_{yy} = \frac{1}{dt}\left[\frac{\overbrace{y + dy - y}^{\text{Length of left edge at } t+dt} - \overbrace{dy}^{\text{Length of left edge at } t}}{dy}\right] = 0 \qquad (7)$$

We get the same result by considering the *right* edge of the fluid particle. Thus both the *x*- and *y*-components of linear strain rate are zero for this flow field.

(*c*) From Eq. 3 we calculate

Linear strain rates: $$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{8}$$

4-59

Solution We are to verify that the given flow field is incompressible using two different methods.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional.

Analysis

(*a*) The volume of the fluid particle at time *t* is.

Volume at time t: $$\mathcal{V}(t) = dxdydz \tag{1}$$

where dz is the length of the fluid particle in the z direction. At time $t + dt$, we assume that the fluid particle's dimension dz remains fixed since the flow is two-dimensional. Thus its volume is dz times the area of the rhombus shown in Fig. P4-58, as illustrated in Fig. 1,

Volume at time t + dt: $$\mathcal{V}(t + dt) = dxdydz \tag{2}$$

Since Eqs. 1 and 2 are equal, the volume of the fluid particle has not changed, and **the flow is therefore incompressible**.

(*b*) We use the equation for volumetric strain rate in Cartesian coordinates, and apply the results of the previous problem,

Volumetric strain rate: $$\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 + 0 + 0 = 0 \tag{3}$$

Where $\varepsilon_{zz} = 0$ since the flow is two-dimensional. Since the volumetric strain rate is zero everywhere, **the flow is incompressible**.

4-60

Solution For the given velocity field we are to calculate the two-dimensional shear strain rate in the x-y plane from fundamental principles and compare with the given equation.

Assumptions **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

Analysis

(a) The shear strain rate is

Shear strain rate in Cartesian coordinates:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (1)$$

From the fundamental definition of shear strain rate in the x-y plane, we consider the bottom edge and the left edge of the fluid particle, which intersect at 90° at the lower left corner at time t (Fig. P4-58). We define angle α between the lower edge and the left edge of the fluid particle, and angle β, the complement of α (Fig. 1). The rate of decrease of angle α over time interval dt is obtained from application of trigonometry. First, we calculate angle β,

Angle β at time $t + dt$:
$$\beta = \arctan\left(\frac{bdydt}{dy}\right) = \arctan\left(bdt\right) \approx bdt \qquad (2)$$

The approximation is valid for very small angles. As the time interval $dt \to 0$, Eq. 2 is correct. At time $t + dt$, angle α is

Angle α at time $t + dt$:
$$\alpha = \frac{\pi}{2} - \beta \approx \frac{\pi}{2} - bdt \qquad (3)$$

During this time interval, α changes from 90° ($\pi/2$ radians) to the expression given by Eq. 2. Thus the rate of change of α is

Rate of change of angle α:
$$\frac{d\alpha}{dt} = \frac{1}{dt}\left[\underbrace{\left(\frac{\pi}{2} - bdt\right)}_{\alpha \text{ at } t+dt} - \underbrace{\frac{\pi}{2}}_{\alpha \text{ at } t}\right] = -b \qquad (4)$$

Finally, since shear strain rate is defined as *half* of the rate of *decrease* of angle α,

Shear strain rate:
$$\varepsilon_{xy} = -\frac{1}{2}\frac{d\alpha}{dt} = \boxed{\frac{b}{2}} \qquad (5)$$

(b) From Eq. 1 we calculate

Shear strain rate:
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(b + 0\right) = \boxed{\frac{b}{2}} \qquad (6)$$

Both methods for obtaining the shear strain rate agree (Eq. 5 and Eq. 6).

4-61

Solution For the given velocity field we are to calculate the two-dimensional rate of rotation in the x-y plane from fundamental principles and compare with the given equation.

Assumptions **1** The flow is incompressible. **2** The flow is steady. **3** The flow is two-dimensional.

Analysis

(a) The rate of rotation in Cartesian coordinates is

Rate of rotation in Cartesian coordinates:
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (1)$$

From the fundamental definition of rate of rotation in the x-y plane, we consider the bottom edge and the left edge of the fluid particle, which intersect at 90° at the lower left corner at time t (Fig. P4-58). We define angle β in Fig. 1, where β is the negative of the angle of rotation of the left edge of the fluid particle (negative because rotation is mathematically positive in the *counterclockwise* direction). We calculate angle β using trigonometry,

Angle β at time $t + dt$:
$$\beta = \arctan\left(\frac{bdydt}{dy}\right) = \arctan\left(bdt\right) \approx bdt \qquad (2)$$

The approximation is valid for very small angles. As the time interval $dt \to 0$, Eq. 2 is correct.

Meanwhile, the bottom edge of the fluid particle has not rotated at all. Thus, the average angle of rotation of the two line segments (lower and left edges) at time $t + dt$ is

Average angle of rotation at time $t + dt$:
$$AVG = \frac{1}{2}\left(0 - \beta\right) \approx -\frac{b}{2}dt \qquad (3)$$

Thus the average rotation *rate* during time interval dt is

Rate of rotation in x-y plane:
$$\omega_z = \frac{d\left(AVG\right)}{dt} = \frac{1}{dt}\left(-\frac{b}{2}dt\right) = \boxed{-\frac{b}{2}} \qquad (4)$$

(b) From Eq. 1 we calculate

Rate of rotation:
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - b\right) = \boxed{-\frac{b}{2}} \qquad (5)$$

Both methods for obtaining the rate of rotation agree (Eq. 4 and Eq. 5).

4-67

Solution We are to calculate the vorticity of fluid particles in a tank rotating in solid body rotation about its vertical axis.

Assumptions **1** The flow is steady. **2** The z-axis is in the vertical direction.

Analysis Vorticity $\vec{\zeta}$ is twice the angular velocity $\vec{\omega}$. Here,

Angular velocity:
$$\vec{\omega} = 360\frac{\text{rot}}{\text{min}}\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{\text{rot}}\right)\vec{k} = 37.70\vec{k}\text{ rad/s} \quad (1)$$

where $\vec{k}$ is the unit vector in the vertical (z) direction. The vorticity is thus

Vorticity:
$$\vec{\zeta} = 2\vec{\omega} = 2\times 37.70\vec{k}\text{ rad/s} = 75.4\vec{k}\text{ rad/s} \quad (2)$$

4-74

Solution We are to determine if the flow is rotational, and if so calculate the z-component of vorticity.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the x-y plane.

Analysis The velocity field is given by

Velocity field, Couette flow:
$$\vec{V} = (u, v) = \left(V\frac{y}{h}\right)\vec{i} + 0\vec{j} \quad (1)$$

If the vorticity is non-zero, the flow is rotational. So, we calculate the z-component of vorticity,

z-component of vorticity:
$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{V}{h} = -\frac{V}{h} \quad (2)$$

Since vorticity is non-zero, **this flow is rotational**. Furthermore, the vorticity is negative, implying that **particles rotate in the clockwise direction**.